MULTISTAGE LOT SIZING PROBLEMS VIA RANDOMIZED ROUNDING

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We study the classical multistage lot sizing problem that arises in distribution and inventory systems. A celebrated result in this area is the 94% and 98% approximation guarantee provided by power-of-two policies. In this paper, we propose a simple randomized rounding algorithm to establish these performance bounds. We use this new technique to extend several results for the capacitated lot sizing problems to the case with submodular ordering cost. For the joint replenishment problem under a fixed base period model, we construct a 95.8% approximation algorithm to the (possibly dynamic) optimal lot sizing policy. The policies constructed are stationary but not necessarily of the power-of-two type. This shows that for the fixed based planning model, the class of stationary policies is within 95.8% of the optimum, improving on the previously best known 94% approximation guarantee.

1. INTRODUCTION

In this paper, we consider multiproduct lot sizing problems that arise in distribution and assembly systems. There is a set N of components. For each component $j \in N$, there is a set π_j (called predecessors of component j) of subcomponents consumed in producing component j. We define the component network G to be a directed network with node set N and arc set $A = \{(i, j) : \ i \in \pi_j\}$. The nodes in G correspond to stages in the assembly process. In other words, the network G corresponds to the flow of materials in the system and contains no circuit. The *final products* in this case correspond to components from the last assembly (or distribution) stage, and hence are produced by the *sinks* in the component network G. Note that a sink in a directed graph refers to node with out-degree of zero. External demands are present only for the final products and are assumed to be constant with rate d_i for item at stage i. $d_i = 0$ if stage i does not correspond to a sink.

Clearly, to satisfy the demand, orders should be placed for the components dynamically in time. If an order is placed for component i, an ordering cost K_i is incurred. Moreover, an incremental echelon holding cost h_i is incurred per unit time the item spends in inventory. The assembly rate is assumed to be infinite. The objective is to schedule orders for each of the components over an infinite horizon so as to minimize *long-run average cost*. We do not allow stockouts in the model. In the rest of the paper, we refer to this problem as (P_{MS}).

Because the optimal dynamic policy can be very complicated, the research community (see for instance Roundy 1985, 1986, Jackson et al. 1985., Muckstadt and Roundy 1993) has focused on the class of stationary and nested policies defined as follows. Orders are placed periodically in time at equal intervals, for each of the components in the system (i.e., stationary policies). If component j is used in the assembly process of component i, then an order is placed for component j only when an order is placed for component i at the same time (i.e., nested policies). Note that because the assembly rate is assumed to be infinite, the assembly lead time is zero, and hence, without loss of generality we can assume that an order is placed for item i only when the inventory level of that item drops to zero. This is known as the *zero ordering property* of the system. Under a stationary and nested policy, the objective is to decide the period T_i that an order is placed. The reason stationary and nested policies are attractive is that they are easy to implement. Muckstadt and Roundy (1993) also discuss in detail the rationale of using order intervals T_i as variables.

Let T_j denote the ordering interval for the items at node $j \in N$. In addition to the set of predecessors π_j of node j, we introduce the set S_j of successors of node j. Furthermore, without loss of generality, we may assume that to assemble a unit item at node j, we consume a unit item from all the predecessors in π_j. We also assume that the holding cost at node j is not bigger than the holding cost at all its predecessors in π_j combined, i.e., we have $h_j \geqslant \sum_{k \in \pi_j} h_k$. Starting from the sinks of the production/distribution system, i.e., from nodes i with $S_i = \varnothing$, we can define recursively the aggregate demand at node j by $D_j = d_j + \sum_{k \in S_j} D_k$. The average inventory and ordering cost structure under stationary and nested policies, derived

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in Roundy (1985, 1986), is as follows:

$$\sum_{i\in N}\left(\frac{K_i}{T_i}+H_i^eT_i\right),$$

with $H_i^e = (h_i - \sum_{j\in\pi_i} h_j)D_i/2$. The term K_i/T_i corresponds to the average ordering cost, and the term $H_i^eT_i$ correspond to the average inventory cost.

Jackson et al. (1985), and Roundy (1985, 1986) consider the following convex relaxation of the problem

$$(P_R)\ Z_R = \min\sum_{i\in N}\left(\frac{K_i}{T_i}+H_i^eT_i\right)$$
$$\text{s.t.}\quad T_i \geqslant T_j \text{ if } (i,j)\in A,$$
$$T_i \geqslant T_L \text{ for each } i.$$

T_L in the above model refers to the fixed planning period, and it can be assumed to be fixed, or a variable to be jointly optimized. Notice that the constraints $T_i \geqslant T_j$ is a relaxation of the condition that policies are nested.

As the objective function is convex, the relaxation (P_R) can be solved in polynomial time using interior point algorithms (see for instance, Nesterov and Nemirovski 1994). For systems with special structure, the running time can be improved substantially. For instance, if G is a tree, Jackson and Roundy (1991) show that the relaxed problem can be solved in $O(n\log n)$ time where $n=|N|$. When G corresponds to a star graph, Queyranne (1987), and Lu and Posner (1994) showed that the relaxed problem can be solved in $O(n)$ time, using a linear time median finding algorithm.

Regarding approximation algorithms, Roundy (1985, 1986), and Maxwell and Muckstadt (1985), in a series of influential papers, showed how to round an optimal solution of the relaxed problem (P_R) to near optimal nested policies. The policies constructed are called *power-of-two policies*, where each T_i is of the form $2^{p_i}T_L$, where p_i is integer. Let Z_H be the value of the heuristic used. They obtained the following bounds:

1. If T_L is not fixed, but subject to optimization, then

$$\frac{Z_H}{Z_R} \leqslant \frac{1}{\sqrt{2}\log 2} \approx 1.02.$$

2. If T_L is fixed, then

$$\frac{Z_H}{Z_R} \leqslant \left(\sqrt{2}+\frac{1}{\sqrt{2}}\right)/2 \approx 1.06.$$

In both cases the bounds are tight. These results are often referred in the literature as 98% (=1/1.02) and 94% (=1/1.06) effective lot sizing policies, respectively.

Roundy (1985) applied the rounding technique to the one warehouse, multiretailer problem (OWMR). In this problem G is a star graph, with the center node representing a warehouse and the leaf nodes representing retailers. The retailers place their orders with the warehouse which in turn orders from an external supplier. Roundy (1985) shows that the optimal policy need not be nested. In fact, the optimal nested policy can be very bad for this class of problems. See Muckstadt and Roundy (1993) for an example. Roundy (1985) studied the class of integer ratio policies, where the ordering intervals of the retailers are restricted to be integer multiples of the ordering interval of the warehouse, or vice versa. He proposed similar 94% (and resp. 98%) effective algorithms to construct efficient integer-ratio policies for the fixed base period (resp. variable base) model. He also shows that these bounds apply even with respect to the *optimal* (possibly *dynamic*) lot sizing policies because the lower bound obtained from the integer-ratio assumption is a valid lower bound for the optimal policies. Finding the optimal lot sizing policies for these problems, though, is still open. It is not known whether these problems can be solved to optimality in polynomial time or whether the problem is NP-hard.

These results have been extensively studied and extended to other versions of lot sizing problems: finite assembly rates (Atkins et al. (1992)), individual capacity bounds of the form $2^{l_i}T_L \leqslant T_i \leqslant 2^{u_i}T_L$, but more general cost structures (Zheng 1987, Fedegruen and Zheng 1993), and backlog (Atkins and Sun 1995). All these extensions use deterministic rounding to generate power-of-two policies with the same 94% and 98% bounds. Jackson et al. (1988) obtained efficien heuristics for the general capacitated version (with bounds of the type $\sum_i T_i \leqslant C$), using the structural results for the uncapacitated version. However, no worst-case bound is known for their heuristics, and there is no guarantee of feasibility of the solution produced by the heuristic. For the case with a linear ordering cost, and with a single capacity constraint, Roundy (1989) obtained a 94% efficient heuristic, using an ingenious rounding heuristic.

A special case of the above model is the classical Joint Replenishment Problem (JRP). In an inventory system of multiple items, cost savings can be obtained when the replenishment of several items are coordinated. Each time an order is placed, a major (joint) ordering cost is incurred, independent of the number of items ordered. A minor ordering cost is also incurred whenever an item is included in an order. This problem is equivalent to the multistage lot sizing problem when the component network G is a star graph, with $V(G)=\{0,1,2,\dots,n\}$ and $E(G)=\{(0,1),\dots,(0,n)\}$. The node 0 represents the major ordering cost. Demand for item i occurs at a continuous constant rate of d_i. Let K_i denote the minor ordering cost. The inventory cost is charged at a rate of h_i. Let K_0 denote the major ordering cost, and $H_i = h_id_i/2$. Then a simple lower bound to the optimal solution is

$$(P_{\text{JRP}})Z_{\text{JRP}} = \min\sum_{i=1}^{n}\left(\frac{K_i}{T_i}+H_iT_i\right)+\frac{K_0}{T_0}$$
$$\text{s.t.}\quad T_i \geqslant T_0 \text{ if } i=1,2,\dots,n,$$
$$T_i \geqslant T_L \text{ for each } i=0,1,2,\dots,n.$$

An efficient approximation algorithm is known for the joint replenishment problem by rounding off the policies obtained from the above relaxation (cf. Muckstadt and Roundy 1993). The main insight from these studies is the surprising effectiveness of the class of stationary policies, which can be shown to be within 94% (resp. and 98%) of the optimum for the fixed base (resp. variable base) model.

Lu and Posner (1994) initiated the study of finding improved worst case guarantees for these problems. For the variable fixed base period model, they show that an $(1+\varepsilon)$-approximation algorithm can be constructed for the joint replenishment and one-warehouse, multiretailer system, where the running time of the algorithm is a polynomial function of $1/\varepsilon$. Their approach, however, fails to produce any improvement for the fixed base planning model because of the discrete nature of the problem (i.e., all ordering intervals have to be a fixed multiple of a based planning interval).

In recent years, there have been many advances in the area of approximation algorithms (cf. Arora 1998). Many notoriously hard discrete optimization problems can now be approximated to within a reasonable guarantee. Many of these advances are obtained via finding a good (fractional) relaxation to the underlying discrete problem, and devising a suitable rounding mechanism on the optimal fractional solution to obtain a feasible solution to the original problem. The analysis of the quality of the solution obtained is aided by an averaging argument by purposely introducing a randomization mechanism into the rounding step, so the quality of the solution obtained can be analyzed via a simple probabilistic computation. Indeed, Roundy's original analysis of the 98% approximation algorithm for the lot sizing problem uses a similar averaging argument. However, he did not elaborate on the generality of this approach to the analysis of other classes of lot sizing problems. In this paper, we show the generality of this approach by introducing a new randomized rounding analysis for the multistage lot sizing problem. By using an improved formulation for the JRP under the fixed base planning period, we also obtain improved approximation results for this class of problems. Our contributions in this paper are mainly as follows:

1. We motivate the idea of randomized rounding by first proposing new 94% and 98% randomized rounding algorithms for both fixed and variable based period models for the multistage lot sizing problems. The purpose of this discussion is to illustrate the simplicity with which the approximation results can be derived and to pave the way for the more complicated proof for the improvement to the joint replenishment problem.

2. Our results generalize immediately to several other extensions considered in the literature. For resource constrained problems under *submodular* ordering cost functions, we propose a rounding procedure that obtains a bound of 1.44 for multiple resource constrained problems, and 1.06 for single-resource constrained problems. These extend the results of Roundy (1989) for the linear ordering cost case.

3. For the JRP under the fixed base period model, we propose an improved 95.8% approximation algorithm. The best known bound prior to this work is the 94% guarantee using the power-of-two policies. More importantly, the bound is valid with respect to the possibly dynamic optimal lot sizing policies. The improvement is obtained using a new and improved relaxation of the problem.

Because our main objective is to demonstrate the simplicity of the randomized rounding methods, we focus on the analysis of the quality of the relaxations and do not discuss the details of solving these relaxations efficiently in practice. In the next section, we establish the well known bound of 94% and 98% using the new randomized rounding idea. In §3, we study the capacitated lot sizing problem with submodular ordering cost. In §4, we describe the improved approximation algorithm for JRP under the fixed order period model.

The technique can be used on a variety of other lot sizing problems (one warehouse multiretailer system for example) to obtain improved approximation guarantee under the fixed base planning period model when restricted to stationary ordering policy. We refer the readers to Teo and Bertsimas (1996) and Teo (1996) for details.

2. RANDOMIZED ROUNDING AND LOT SIZING PROBLEMS

In this section, we introduce the key randomized rounding ideas in the context of nested policies. To see how our rounding algorithm departs from the classical one, we first review the basic idea behind the classical approach.

Consider the single-item inventory lot sizing problem. The graph G in this case is simply a singleton. It is well known that the optimal solution to this problem satisfies the following property:

$$\frac{K}{T^*} = HT^* \text{ where } T^* \text{ is the optimal solution.} \tag{1}$$

This formula is the well-known economic order quantity (EOQ) solution to the problem. The optimal solution T^* is surprising elastic because if $T = \lambda T^*$, then the deviation from optimal cost is just $(\lambda + \frac{1}{\lambda})/2$. For $\lambda = \sqrt{2}$ or $1/\sqrt{2}$, the bound is 1.06. This property is essentially the basis for the 94% and 98% guarantee of the power-of-two policy constructed by Roundy (1985, 1986) Jackson et al. (1985). These papers construct a power-of-two policy that rounds each T_i^* to T_i^o with $T_i^o = 2^{p_i} T_L$ for some p_i, and T_L is either fixed or optimally selected. By a suitable choice of p_i, we can ensure that T_i^0 lies in the interval

$$\left[\frac{T_i^*}{\sqrt{2}}, \sqrt{2}T_i^*\right].$$

Furthermore, by studying the structure of the optimal solution, the previous papers essentially established that for the more complicated model (P_R), the optimal solution satisfies an EOQ type property with the same elastic structure in the optimal solution. Since $T_i^*/\sqrt{2} \leqslant T_i^o \leqslant \sqrt{2}T_i^*$, the 94% bound holds immediately. By optimizing the choice of T_L, the bound can be improved to 98%.

The above rounding technique works, however, only when the policy T^* used in the rounding procedure is the optimal solution to the convex programming relaxation, as the argument depends critically on the EOQ structure in Equation (1). The 94% bound does not apply if T^*, the input to the rounding process, is simply a feasible solution to (P_R), even though T^* might be close to optimal.

2.1. Fixed Base Period Model: 94% Approximation Algorithm

Consider the relaxation

$$(P_R)\ Z_R = \min \sum_{i\in N}\left(\frac{K_i}{T_i} + H_i^e T_i\right)$$
$$\text{s.t.}\quad T_i \geqslant T_j \text{ if } (i,j)\in A,$$
$$T_i \geqslant T_L \text{ for each } i.$$

As the policies constructed from the relaxation may not satisfy the nested property, the average inventory cost structure might be complicated. As in Roundy (1985, 1986), we round the solution to one that satisfies the power-of-two property. Roundy showed that the optimal solution to (P_R) has similar structure as in the simple EOQ model (cf. Equation (1)), and hence the 1.06 bound holds also for the problem (P_R). We perform a randomized rounding analysis and show that the dependence on the structure of the optimal solution can be removed for the 1.06 bound to hold. Thus this analysis allows us to analyze more complicated lot sizing models, without having to characterize the structure of the optimal solutions.

Consider the following randomized rounding algorithm:

ALGORITHM A.

1. Let T_L be fixed. Let $T = (T_1, \dots, T_n)$ be a feasible solution to relaxation (P_R).
2. Write $T_i = 2^{p_i} z_i T_L$, where $1 \leqslant z_i < 2$.
3. Generate a point Y in the interval $[1,2]$, with probability distribution

$$P(Y\leqslant y) = F(y) = \frac{y^2-1}{1+\frac{y^2}{2}}.$$

4. For all i, if $z_i < Y$, then $T_i^o = 2^{p_i}T_L$; otherwise $T_i^o = 2^{p_i+1}T_L$.

Note that the above rounding scheme always generates a nested solution $(T_1^o, T_2^o, \dots, T_n^o)$ the solution obtained is of the power-of-two type.

THEOREM 1. *Let $T = (T_1, \dots, T_n)$ be any feasible solution to Problem (P_R) with cost $G(T) = \sum_{i\in N}(\frac{K_i}{T_i} + H_i^e T_i)$. Algorithm A returns a power-of-two policy $T^o = (T_1^o, T_2^o, \dots, T_n^o)$ with expected cost at most* $1.06\ G(T)$.

PROOF. It is easy to see that

$$\begin{aligned} E(T_i^o) &= 2^{p_i}T_L P(z_i < Y) + 2^{p_i+1}T_L P(z_i \geqslant Y)\\ &= 2^{p_i}T_L(1-F(z_i)) + 2^{p_i+1}T_L F(z_i)\\ &= T_i[1+F(z_i)]/z_i, \end{aligned}$$

i.e.,

$$E(T_i^o) = T_i \frac{3z_i}{z_i^2+2} \leqslant \frac{\sqrt{2}+1/\sqrt{2}}{2} T_i \approx 1.06\, T_i.$$

The bound follows because the maximum value of the function $\frac{3z_i}{z_i^2+2}$ is at most $3\sqrt{2}/4$.

Similarly, $E(1/T_i^o) = (1-F(z_i))/(2^{p_i}T_L) + F(z_i)/(2^{p_i+1}T_L) = (1-F(z_i)/2)z_i/T_i$, i.e.,

$$E\left(\frac{1}{T_i^0}\right) = \frac{1}{T_i}\frac{3z_i}{z_i^2+2} \leqslant \frac{\sqrt{2}+1/\sqrt{2}}{2}\frac{1}{T_i}.$$

The theorem follows as $E(G(T^o)) \leqslant 1.06 G(T)$. □

Note that the distribution function $F(y)$ is chosen so that $(1+F(y))/y = y(1-F(y)/2) = 3y/(y^2+2)$. The maximum is attained at the point $y=\sqrt{2}$ with a value of $3\sqrt{2}/4 \approx 1.06$. Furthermore, using the optimal solution to (P_R) as input to the rounding process, we obtain $E(G(T^o)) \leqslant 1.06 Z_R$, which is a $1/1.06 \approx 94\%$ approximation algorithm to the original lot sizing problem.

DERANDOMIZATION. The above randomized algorithm can be made deterministic: Without loss of generality, assume that the z_i's are in non-decreasing order, i.e., $z_1 \leqslant z_2 \leqslant \cdots \leqslant z_n$. For all y in $[z_i, z_{i+1})$, the randomized algorithm returns the same solution. Hence, there are at most $n+1$ distinct solutions obtained from the randomized rounding procedure. These distinct solutions can be obtained in a deterministic manner, once the z_is are sorted. Thus, the best solution can be obtained in time $O(n \log n)$.

2.2. Variable Base Period Model: The 98% Approximation Algorithm

The same insensitivity result can also be improved to a 98% guarantee, if one allows the base period T_L to vary, i.e. with T_L as a variable in (P_R). In fact, Roundy's 98% algorithm (1985, 1986) already has this feature. We recast Roundy's algorithm into the following randomized rounding algorithm:

ALGORITHM B.

1. Let $T = (T_1, \dots, T_n, T_L)$ be a feasible solution to (P_R), with $T_L > 0$.
2. Let $T_i = 2^{p_i}T_L z_i$, where $1 \leqslant z_i < 2$.
3. Generate a point Y in the interval $[1,2]$, with probability distribution

$$F(y) = \frac{\log y}{\log 2}.$$

4. For all i, if $Y > z_i$, then $T_i^o = 2^{pi}\frac{Y}{\sqrt{2}}$; otherwise $T_i^o = 2^{pi+1}\frac{Y}{\sqrt{2}}$. Let $T_L = \frac{Y}{\sqrt{2}T_L}$.

Note that T_i^o lies in the interval $[\frac{T_1}{\sqrt{2}}, \sqrt{2}T_i]$. This property is useful when we are dealing with capacitated systems (see §3). Furthermore, it is clear that $(T_1^o, T_2^o, \dots, T_n^o, T_L^o)$ is nested. Note that $dF(y)/dy = 1/(y\log 2)$.

Theorem 2. *Let* $(T_1, \ldots, T_n, T_L)$ *be any feasible solution to* (P_R) *with cost* $G(T)$. *Algorithm B returns a power-of-two policy* $(T_1^o, T_2^o, \ldots, T_n^o, T_L^o)$ *with expected cost at most* $\frac{G(T)}{\sqrt{2}\log(2)} \approx 1.02\, G(T)$.

Proof. Without loss of generality, we may assume $T_L = 1$. Then,

$$E(T_i^o) = \int_1^{z_i} 2^{p_i+1}\frac{y}{\sqrt{2}}\frac{1}{y\log 2}dy + \int_{z_i}^2 2^{p_i}\frac{y}{\sqrt{2}}\frac{1}{y\log 2}dy$$
$$= \frac{2^{p_i}}{\sqrt{2}\log 2}\left(\int_1^{z_i} 2dy + \int_{z_i}^2 dy\right)$$
$$= \frac{2^{p_i}[2(z_i-1)+(2-z_i)]}{\sqrt{2}\log 2} = \frac{T_i}{\log 2\sqrt{2}}.$$

Similarly,

$$E(1/T_i^o) = \frac{\sqrt{2}\int_1^{z_i} 2^{-p_i-1}(1/y^2)dy + \sqrt{2}\int_{z_i}^2 2^{-p_i}(1/y^2)dy}{\log 2}$$
$$= \frac{\sqrt{2}2^{-p_i}(1/2 - \frac{1}{2z_i} - 1/2 + \frac{1}{z_i})}{\log 2} = \frac{1}{T_i \log 2\sqrt{2}},$$

and the theorem follows as $1/(\sqrt{2}\log 2) \approx 1.02$. □

Derandomization. Suppose $z_1 \leqslant z_2 \leqslant \cdots \leqslant z_n$. For y in $[z_i, z_{i+1})$, suppose the algorithm returns a policy with cost $A/y + By$,then for all other y' in the same interval, the algorithm returns a policy with cost $A/y' + By'$. By choosing a y' in the interval that minimizes this term, and doing the same for each interval partitioned by the z_is, we obtain an $O(n\log n)$ deterministic algorithm, which is exactly Roundy's rounding procedure.

The argument used above can easily be adapted to analyze more complicated objective cost functions. For instance, we have the following:

Theorem 3. *Under Algorithm B*,

$$E\Big(\big(T_i^o\big)^2\Big)/T_i^2 = E\left(\frac{1}{T_i^o}\right)^2 = \frac{3}{4\log(2)} \approx 1.082;$$
$$\frac{1}{\sqrt{2}\log(2)} \leqslant E\left(\frac{1}{T_i^o T_j^o}\right)T_i T_j \leqslant \frac{3}{4\log(2)};$$
$$\frac{1}{\sqrt{2}\log(2)} \leqslant E(T_i^o T_j^o)\frac{1}{T_i T_j} \leqslant \frac{3}{4\log(2)};$$
$$E\left(\frac{T_i^o}{T_j^o}\right) \leqslant 1.06\frac{T_i}{T_j}.$$

Theorem 3 will be used later to analyze a capacitated version of the lot sizing model.

2.3. Submodular Ordering Costs

In the above models, the ordering cost is determined by the set of items ordered in a linear fashion. For most realistic assembly environments, a fixed charge K_0 is incurred whenever an order is placed, regardless of the number of items ordered at the time. This gives rise to an ordering cost $K(S) \equiv K_0 + \sum_{i\in S} K_i$ for a set S of items that are ordered at the same instance. More generally, when $K(\cdot)$ is submodular and nondecreasing, Federgruen et al. (1992) and Zheng (1987) have shown that the corresponding lot sizing problem has average cost bounded below by

$$G'(T) \equiv \max_{k=(k_1,\ldots,k_n)} \sum_j k_j/T_j + H_j^e T_j,$$

where $T = (T_1, \ldots, T_n)$ is feasible lot sizing policy, and the vector $k = (k_1, \ldots, k_n)$ ranges over the polymatroid

$$\mathcal{P} = \left\{k\colon \sum_{j\in S} k_j \leqslant K(S), \sum_{j\in N} k_j = K(N), k_j \geqslant 0\right\}.$$

Note that $G'(T)$ is a convex function in T. Furthermore, if T is a power-of-two policy, then the average lot sizing cost is exactly $G'(T)$. A lower bound to the lot sizing problem, under a submodular ordering cost function, can be obtained by solving

$$(P_{\text{SUB}})\ Z_{\text{SUB}} = \min_T \max_k \sum_{j\in N}\left(\frac{k_i}{T_i} + H_i^e T_i\right)$$
$$T_i \geqslant T_j \text{ if } (i,j) \in A,$$
$$T_i \geqslant T_L \text{ for each } i,$$
$$k \in \mathcal{P}.$$

For each fixed T, it is well known from the work of Edmonds (1970) that the solution K^* that maximizes $G'(T)$ can be obtained by a greedy algorithm. The greedy procedure sorts the indices in decreasing order of $\frac{1}{T_i}$ and selects the values for k_i^* greedily in that order. Hence if T^1 is another lot sizing policy that preserves the order of T, i.e., $T_{i_1}^1 \leqslant T_{i_2}^1 \leqslant \cdots \leqslant T_{i_n}^1$ and $T_{i_1} \leqslant T_{i_2} \leqslant \cdots \leqslant T_{i_n}$ for some ordering of the indices, and if k^* maximizes $G'(T)$, then K^* is also an optimal solution to $G'(T^1)$.

Let (K^*, T^*) be an optimal solution to (P_{SUB}). Let T^o be the solution obtained by rounding T^* using Algorithm A or B. Since the randomized rounding algorithm preserves the order of the original solution, it follows that $E(G'(T^o)) = E(\sum_j k_j^*/T_j^o + H_i^e T_i^o)$. Furthermore, T^o is a power-of-two policy. In this way, Algorithms A and B can be used to round the fractional optimal solution in (P_{SUB}) to 94% and 98% optimal power-of-two solutions.

Theorem 4. *Let* $(T_1, \ldots, T_n, T_L)$ *be any feasible solution to* (P_{SUB}) *with cost* $G'(T)$, *Algorithm A returns a power-of-two policy (with fixed base* T_L*) with an expected cost of not more than* 1.06 $G'(T)$. *Similarly, Algorithm B returns a power-of-two policy* $(T_1^o, T_2^o, \ldots, T_n^o, T_L^o)$ *with expected cost at most* 1.02 $G'(T)$.

3. RESOURCE CONSTRAINED LOT SIZING PROBLEMS

For the lot sizing model (P_{MS}) with resource constraints of the type

$$\sum_j a_{ij}/T_j \leqslant A_i,\ i = 1, \ldots, m \tag{2}$$

added, Roundy (1989) shows that there is a power-of-two policy (variable base) with cost at most 1.44 times the optimal solution. We generalize this result to the lot sizing problems with submodular joint ordering cost function. Consider the following procedure.

Algorithm C.

1. Let (K^*, T^*) be an optimal solution to (P_{SUB}) *with the resource constraints* (2) *added*.

2. Use $T_j = \sqrt{2}T_j^*$ for all j in Algorithm B to obtain a power-of-two policy T^o.

Note that in Algorithm B, $T_i^o \geqslant \frac{T_i}{\sqrt{2}}$, and hence $T_i^o \geqslant T_i^*$. Hence, (T_j^o) satisfies the resource constraints as (T_j^*) satisfies (2). Also, because scaling by $\sqrt{2}$ does not affect the ordering of T_j^*, the result follows directly from

$$E(T_j^o) \leqslant \frac{1}{\sqrt{2}\log(2)} T_j = \frac{1}{\sqrt{2}\log(2)}\sqrt{2}T_j^* \approx 1.44 T_j^*,$$

and

$$E\left(\frac{1}{T_j^o}\right) \leqslant \frac{1}{\sqrt{2}\log(2) T_j} \leqslant \frac{1}{T_j^*}.$$

The solution K^* is also a maximum solution to $G'(T^o)$. Hence, we have Theorem 5.

Theorem 5. *Let T^* be an optimal solution to the resource constrained version of (P_{SUB}). Algorithm C return a power-of-two policy with expected cost at most* 1.44 *times of the optimal cost.*

3.1. Single-Resource Constraint

In the rest of this section, we show that for the single-resource constrained problem, the bound can be improved to 1.06. We prove that bound for submodular ordering cost case, generalizing a result obtained first by Roundy (1989) for the resource constrained version of (P_{MS}). This problem is interesting, since it constitutes a valid relaxation for the well-known economic lot scheduling problem. See Dobson (1987) and Roundy (1989) for a review of this model and its connection with the economic lot scheduling problem.

Consider the following algorithm.

Algorithm D.

1. Let T^* be the solution obtained from single resource constraint of the type $\sum_j a_j/T_j \leqslant A$ added to (P_{SUB}).

2. Use Algorithm B with T^* to obtain T^o. Let $\alpha = \max(1, \sum_j a_j/(T_J^o A))$, where $A = \sum_j a_j/T_i^*$.

3. Use αT^o as the lot sizing solution.

It can be shown that the above algorithm reduces to Roundy's (1989) algorithm, though the initial policy T^o before the scaling is different. Furthermore, Roundy (1989) contains a more careful choice of α to improve the policy, but it does not seem to be useful in improving the worst-case bound.

Theorem 6. *Algorithm D returns a power-of-two policy αT^o which is within* 1.06 *from the optimal cost.*

Proof. We note that scaling by α does not alter the order T^o. Hence, αT^o has the same order as T^o and T^*. Let $K^* = (k_1^*, \ldots, k_n^*)$ be optimal solution to $G'(T^*)$.

$$E(G'(\alpha T^o)) = E\left(\sum_j \frac{k_j^*/T_j^o}{\alpha}\right) + E\left(\sum_j H_j T_j^o \alpha\right).$$

If $\alpha = 1$, then the bound follows from Theorem 2. Consider the case $\alpha = \sum_j a_j/(AT_j^o)$. We denote by $E\alpha(\cdot)$ the conditional expectation $E(\cdot|\alpha = \sum_j a_j/(AT_j^o))$. Let $X = T_j^*/T_j^o$ with probability $a_j/(T_j^* A)$ for all j. We have

$$E_\alpha(X) = \sum_j \frac{T_j^*}{T_j^o}\frac{a_j}{T_j^* A} = \alpha.$$

By Jensen's inequality, $E_\alpha(1/X) \geqslant 1/E_\alpha(X)$, and therefore

$$\frac{1}{\alpha} \leqslant \sum_j \frac{a_j/T_j^*}{A}\frac{T_j^o}{T_j^*}.$$

So,

$$\begin{aligned}
E_\alpha(G'(\alpha T^o)) &= E_\alpha\left(\sum_j \frac{k_j^*}{\alpha T_j^o}\right) + E_\alpha\left(\sum_j H_j T_j^o \alpha\right)\\
&\leqslant E_\alpha\left(\left(\sum_j \frac{k_j^*}{T_j^o}\right)\left(\sum_i \frac{a_i/T_i^*}{A}\frac{T_i^o}{T_i^*}\right)\right)\\
&\quad + E_\alpha\left(\left(\sum_j H_j T_j^o\right)\left(\sum_i \frac{a_i}{A}\frac{1}{T_i^o}\right)\right)\\
&\qquad \text{(by Jensen's inequality)}\\
&\leqslant E_\alpha\left(\sum_{i,j} \frac{k_j^* a_i/T_i^*}{A}\frac{T_i^o}{T_i^* T_j^o}\right) + E_\alpha\left(\sum_{i,j}\frac{a_i H_j}{A}\frac{T_j^o}{T_i^o}\right).
\end{aligned}$$

By Theorem 3,

$$\begin{aligned}
E_\alpha(G'(T^o\alpha)) &\leqslant 1.06\left(\sum_{i,j}\frac{k_j^* a_i/T_i^*}{A}\frac{1}{T_j^*} + \sum_{i,j}\frac{a_i H_j}{A}\frac{T_j^*}{T_i^*}\right)\\
&= 1.06\left(\left(\sum_j \frac{k_j^*}{T_j^*}\right)\left(\sum_i \frac{a_i/T_i^*}{A}\right)\right.\\
&\qquad \left. + \left(\sum_j H_j T_j^*\right)\left(\sum_i \frac{a_i/T_i^*}{A}\right)\right)\\
&= 1.06\left(\sum_j k_j^*/T_j^* + \sum_j H_j T_j^*\right). \qquad \square
\end{aligned}$$

4. JOINT REPLENISHMENT PROBLEMS

In this section, we focus on an improved approximation algorithm for the joint replenishment problem under the fixed base period model. Recall that each time an order is placed, a joint ordering cost K_0 is incurred, independent of the number of items ordered. An additional ordering cost K_i is also incurred whenever an item is included in an order. The inventory cost is charged at a rate of h_i and $H_i = h_i d_i/2$.

For any t integer, we let

$$L_i(t) \equiv \frac{K_i}{t} + H_i t.$$

If $k < t < k+1$ for some integer k, let

$$L_i(t) = (t-k)L_i(k) + (k+1-t)L_i(k+1).$$

We consider the following relaxation for the JRP:

$$\begin{aligned}(P_{JR}) : Z_{JR} = \min \sum_{i\geqslant 1} L_i(T_i) + \frac{K_0}{T_0} \\ \text{s.t. } T_i \geqslant T_0, \qquad \forall\, i \geqslant 1, \\ T_0 \geqslant 0.\end{aligned}$$

Note that the function $L(t)$ is piecewise linear and convex. Let Z^* denote the optimal solution to the JRP under the fixed base period model, over all possibility dynamic policies.

PROPOSITION 1. *In the optimum replenishment policy, all ordering intervals are bounded by some constant M for some large enough M that depends only on the ordering and holding cost rates of the items.*

PROOF. It is clear that the optimum replenishment policy will place an order for an item if and only if the inventory level of the item drops to zero. This follows from the assumption that replenishement is instantaneous and the lead time is zero. Thus, we only need to specify the ordering interval as this will automatically determine the order quantity. Suppose that for item i, there is an ordering interval of k_i for some integer k_i (note that $T_L = 1$). In the optimum policy, if we replace the single order by k_i smaller orders, each with enough demand to last through a single time unit, the new policy will increase the ordering cost by at most

$$(K_0 + K_i)(k_i - 1),$$

whereas the holding cost of item i decreases by

$$\frac{h_i}{2}k_i^2 d_i - \frac{h_i}{2}k_i d_i \geqslant H_i(k_i-1)^2.$$

By the optimality of the original policy, we must have

$$\begin{aligned}H_i(k_i-1)^2 < (K_0+K_i)(k_i-1), \\ \text{i.e., } k_i < 1 + \frac{K_0+K_i}{H}.\end{aligned}$$

This proves the proposition, with $M = \max_i(1 + \frac{K_0+K_i}{H_i})$. □

PROPOSITION 2. *The value Z_{JR} of Problem (P_{JR}) is a lower bound on the optimal solution value Z^* of the joint replenishment problem under general dynamic policies, i.e.,*

$$Z^* \geqslant Z_{JR}.$$

PROOF. Consider an optimal policy over an interval $[0, t']$. During this interval, numerous orders will be placed for each item. By a slight abuse of notation, we will say that an order is of order interval k if the time period between the order and the next order is k. Let $m_i(k)$ be the number of orders placed for item i with order intervals k. $\sum_k m_i(k)$ is thus the total number of orders placed during $[0, t']$. Let $I_i(t)$ be the inventory level of item i at time t. The total number of distinct orders placed is at least $\max_i \sum_k m_i(k)$. The total joint replenishment cost over $[0, t']$ is thus at least

$$K_0 \max_i\left(\sum_k m_i(k)\right) + \sum_{i\geqslant 1} K_i\left(\sum_k m_i(k)\right) + \sum_{i\geqslant 1}\int_0^{t'} h_i I_i(t)\,dt.$$

Let E_i denote the inventory level of item i at time t'. By Proposition 1, $E_i/d_i \leqslant M$. Note that

$$\int_0^{t'} h_i I_i(t)\,dt = h_i\left(\sum_k m_i(k)\frac{d_i k^2}{2} - \frac{E_i^2}{2d_i}\right).$$

Let, $\epsilon_i(t')$ denote the error term $\frac{E_i^2}{2d_i t'}$. Then,

$$\begin{aligned}&\frac{1}{t'}\left(K_i\left(\sum_k m_i(k)\right) + \int_0^{t'} h_i I_i(t)\,dt\right) \\ &= \left(K_i\frac{\sum_k m_i(k)}{t'} + H_i\frac{\sum_k m_i(k)k^2}{t'}\right) - \epsilon_i(t') \\ &= \left(K_i\frac{\sum_k m_i(k)}{\sum_k m_i(k)k} + H_i\frac{\sum_k m_i(k)k^2}{\sum_k m_i(k)k}\right)\frac{\sum_k m_i(k)k}{t'} - \epsilon_i(t'),\end{aligned}$$

where

$$0 \leqslant \epsilon_i(t') \leqslant \frac{h_i d_i}{2t'}M^2.$$

Note that $\epsilon_i(t') \to 0$ as $t' \to \infty$.

Rewriting the expression, we have

$$\begin{aligned}&\frac{1}{t'}\left(K_i\Big(\sum_k m_i(k)\Big) + \int_0^{t'} h_i I_i(t)\,dt\right) \\ &= \sum_j\left(\frac{K_i}{j}\frac{m_i(j)j}{\sum_k m_i(k)k} + H_i j\frac{m_i(j)j}{\sum_k m_i(k)k}\right) \\ &\quad \times \frac{\sum_k m_i(k)k}{t'} - \epsilon_i(t') \\ &= \left(\sum_j \frac{m_i(j)j}{\sum_k m_i(k)k}L_i(j)\right)\frac{\sum_k m_i(k)k}{t'} - \epsilon_i(t'). \qquad (3)\end{aligned}$$

By the convexity of $L_i(t)$, it follows that

$$\sum_j \frac{m_i(j)j}{\sum_k m_i(k)k}L_i(j) \geqslant L_i\left(\sum_j \frac{m_i(j)j}{\sum_k m_i(k)k}j\right). \qquad (4)$$

Let

$$T_i \equiv \sum_j \frac{m_i(j)j}{\sum_k m_i(k)k}j = \frac{\sum_j m_i(j)j^2}{\sum_k m_i(k)k},$$

and

$$T_0 \equiv \frac{\min_i(\sum_j m_i(j)j)}{\max_i(\sum_k m_i(k))}.$$

Note that

$$T_i = \frac{\sum_j m_i(j) j^2}{\sum_k m_i(k) k} \geqslant \frac{\sum_j m_i(j) j}{\sum_k m_i(k)},$$

as

$$\begin{aligned}\Big(\sum_j m_i(j) j^2\Big)\Big(\sum_k m_i(k)\Big) &= \sum_{j,k} m_i(j) j^2 m_i(k)\\ &= \sum_{j<k} m_i(j)(j^2+k^2) m_i(k)\\ &\geqslant \sum_{j<k} 2jk m_i(j) m_i(k)\\ &= \Big(\sum_k m_i(k) k\Big)\Big(\sum_j m_i(j) j\Big).\end{aligned}$$

It thus follows that $T_i \geqslant T_0$ for all $i > 1$.

Now,

$$\begin{aligned}&\frac{1}{t'}\Big(K_i\Big(\sum_k m_i(k)\Big) + \int_0^{t'} h_i I_i(t)\,dt\Big)\\ &= \Big(\sum_j \frac{m_i(j) j}{\sum_k m_i(k) k} L_i(j)\Big)\frac{\sum_k m_i(k) k}{t'} - \epsilon_i(t') \text{ (from (3))}\\ &\geqslant L_i\Big(\sum_j \frac{m_i(j) j}{\sum_k m_i(k) k} j\Big)\frac{\sum_k m_i(k) k}{t'} - \epsilon_i(t') \text{ (from (4))}\\ &= \frac{\sum_k m_i(k) k}{t'} L_i(T_i) - \epsilon_i(t'),\end{aligned}$$

and

$$K_0 \frac{\max_i(\sum_k m_i(k))}{t'} = K_0 \frac{1}{T_0} \frac{\min_i(\sum_j m_i(j) j)}{t'}.$$

So, we have

$$\begin{aligned}&\frac{1}{t'}\Big(K_0 \max_i \Big(\sum_k m_i(k)\Big) + \sum_{i\geqslant 1} K_i \Big(\sum_k m_i(k)\Big)\\ &\qquad + \sum_{i\geqslant 1}\int_0^{t'} h_i I_i(t)\,dt\Big)\\ &\geqslant \frac{\min_i(\sum_j m_i(j) j)}{t'}\Big(K_0 \frac{1}{T_0} + \sum_{i\geqslant 1} L_i(T_i)\Big) + \epsilon_i(t')\\ &\geqslant \frac{\min_i(\sum_j m_i(j) j)}{t'} Z_{JR} + \epsilon_i(t').\end{aligned}$$

Let $t' \to \infty$, then $\epsilon_i(t') \to 0$. On the other hand, because all order intervals are bounded by M,

$$t' - M \leqslant \sum_j m_i(j) j \leqslant t',$$

so

$$\frac{\min_i(\sum_j m_i(j) j)}{t'} \to 1.$$

Hence

$$\begin{aligned}Z^* &\geqslant \lim_{t'\to\infty} \frac{1}{t'}\Big(K_0 \max_i \Big(\sum_k m_i(k)\Big) + \sum_{i\geqslant 1} K_i\Big(\sum_k m_i(k)\Big)\\ &\qquad + \sum_{i\geqslant 1}\int_0^{t'} h_i I_i(t)\,dt\Big) \geqslant Z_{JR}.\end{aligned}$$

Therefore, Z_{JR} is a valid lower bound for the optimal replenishment solution. □

Problem (P_{JR}) can be viewed as a convex integer programming problem with separable objective function, and a totally unimodular constraint matrix. It can be solved efficiently using the algorithm proposed by Hochbaum and Shantikumar (1990). In fact, for the improved approximation algorithm, we do not need to utilize the lower bound in its full generality. We need only to ensure that the ordering interval T_i, whenever $T_i \leqslant 3T_L$, is in the set $\{T_L, 2T_L, 3T_L\}$.

Let T^* be an optimal solution to Problem (P_{JR}). The piecewise linear cost structure of $L_i(t)$ ensures that the coordinates T_i^* are integers for all i (assuming $T_L = 1$). Hence the policies in T^* satisfy the fixed base ordering period condition. If the policies T_i^*s are nested, i.e., there exists T_i^* such that T_i^* divides T_j^* for all other j, then this relaxation is exact. We describe next how to round the policies obtained in T^* into a nested policy. The main idea behind the rounding heuristic is as follows.

> Consider the class of power-of-two policies (called Class 1 policies). We observed that the 94% worst-case bound is achieved only if there is some item i with $T_i^* = \sqrt{2}\, 2^{k_i}$ for some integral k_i. For these intervals, rounding off to policies of the Type 1 or $3\cdot 2^{p_i}$ (called Class 2 policies) can be more efficient. However, for the case $kT_i^* = 2$, the Class 2 policies can be ineffective as we are rounding $T_i^* = 2$ to 1 or 3. By properly trading off the two classes of policies and carefully handling the case with $T_i^* = 2$, we can achieve a better guarantee.

Let $p = 0.7$, $q = 0.3$, and let

$$a = 2\sqrt{\frac{p + 3q/2}{p + 2q/3}}, \quad b = 2\sqrt{\frac{2p + 3q/2}{p/2 + 2q/3}},$$

and

$$F(p, z) = \frac{\frac{1}{4}(p + \frac{2}{3}q) z^2 - \frac{3}{2}q - p}{p(1 + \frac{1}{8}z^2)},$$

$$F'(p, z) = \frac{\frac{1}{9}(q + \frac{3}{4}p) z^2 - \frac{4}{3}p - q}{q(1 + \frac{1}{18}z^2)}.$$

Note that $2 < a < b < 4$, and $3 < b < 2a < 6$.

POLICY 1. Let $T_i^* = 2^{p_i} z_i$, where z_i is in the interval $[1, 2)$, and p_i integer. Let Y be a random number generated in the interval $[a, b]$ with distribution function $P(Y \leqslant y) = F(p, y)$. Let

$$T_i^1 = \begin{cases} 2^{p_i}, & \text{if } 2z_i < Y,\\ 2^{p_i+1}, & \text{if } 2z_i \geqslant Y.\end{cases}$$

Note that if $z_i < a/2$ or $z_i > b/2$, then z_i is always rounded to 1 or 2, regardless of the value of Y. So if z_i falls into this range, the rounding is deterministic.

POLICY. For all items with $T_i^* \geqslant 3$, let $T_i^* = 3\cdot 2^{p_i} z_i'$, is in the interval [1, 2). Let Y' be a random number generated in the interval $[b, 2a]$ with distribution function $P(Y' \leqslant y) = F'(p, y)$. Let

$$T_i^2 = \begin{cases} 3 \cdot 2^{p_i}, & \text{if } 3z_i' < Y', \\ 3 \cdot 2^{p_i+1}, & \text{if } 3z_i' \geqslant Y'. \end{cases}$$

Note that if $z_i < b/3$ or $z_i > 2a/3$, then z_i is always rounded to 1 or 2, regardless of the value of Y'. So if z_i falls into this range, the rounding is deterministic.

For all items i with $T_i^* = 2$, we round all of them (in the same manner) to $T_i^2 = 3$ with probability $\frac{9}{14}$ and to $T_i^2 = 1$ with probability $\frac{5}{14}$.

Finally, if $T_i^* = 1$, $T_i^2 = 1$.

Note that in this way, for $T_i^* = 2$,

$$\frac{E(T_i^2)}{2} = \frac{8}{7} = 2E\left(\frac{1}{T_i^2}\right).$$

Also, the functions $F(p, z)$ and $F'(p, z)$ are selected so we have $F(p, a) = 0 = F'(p, b)$, and $F(p, b) = 1 = F'(p, 2a)$. Furthermore, F and F' are nondecreasing and are valid distribution functions.

ALGORITHM E.

1. Select Policy 1 with probability p, and Policy 2 with probability q.
2. Let T^o denote the policy selected.

THEOREM 7. *Algorithm E returns a policy with expected cost at most* $1.043\ Z_{JR} \leqslant 1.043z^*$.

REMARK 1. In essence, the above theorem says that one of the two policies constructed above will attain a bound of at most 1.043. The first policy has the classical power-of-two structure and is therefore easily implemented in practice. The second policy has order intervals of the type $\{1, 3, 6, 12, \dots\}$ and can also be easily implemented in practice.

REMARK 2. In Policy 2, we lose a bit in rounding order intervals of type $T_i^* = 2$ to 1 or 3. The bound can be tightened slightly if the optimal solution to the relaxation does not have order intervals of the type $2T_L$. This difficulty also precludes the possibility of extending the technique to include order intervals of type $5T_L$ and above.

PROOF. If $T_i^* = 2$, then

$$\frac{E(T_i^2)}{2} = 2E\left(\frac{1}{T_i^2}\right) = p + \frac{8}{7}q \approx 1.0428.$$

Thus we need only to consider the case when T_i^* is greater than 3. Suppose T_i^* lies in (a) $[2^{p_i}a, 2^{p_i}b]$ or (b) $[2^{p_i}b, 2^{p_i+1}a]$. In Case (a), Policy 2 always rounds T_i to 3.2^{p_i}, whereas in Case (b), Policy 1 always rounds T_i to 2^{p_i+2}.

Case (a). T_i^* lies in $[2^{p_i}a, 2^{p_i}b]$, i.e., $T_i^* = 2^{p_i}w_i = 2^{p_i+1}z_i$, where $w_i \in [a, b]$ and $z_i \in [1, 2]$. Then

$$\begin{aligned} E(T_i^o) &= qE(T_i^2) + pE(T_i^1) \\ &= q(3.2^{p_i}) + p\left(2^{p_i+1}P(2z_i < Y) + 2^{p_i+2}P(2z_i \geqslant Y)\right) \\ &= T_i^*\left(\frac{3q}{w_i} + p\frac{2(1+F(w_i))}{w_i}\right), \end{aligned}$$

and

$$\begin{aligned} E\left(\frac{1}{T_i}\right) &= qE\left(\frac{1}{T_i^2}\right) + pE\left(\frac{1}{T_i^1}\right) \\ &= \frac{1}{T_i^*}\left(q\frac{w_i}{3} + p\left(1 - \frac{F(w_i)}{2}\right)\frac{w_i}{2}\right). \end{aligned}$$

We have chosen $F(p, \cdot)$ such that

$$q\frac{3}{w_i} + p\frac{2(1+F(w_i))}{w_i} = q\frac{w_i}{3} + p\left(1 - \frac{F(w_i)}{2}\right)\frac{w_i}{2}.$$

With this choice of F, and $p = 0.7$, $q = 0.3$, we can optimize the bound over the range of $w_i \in [a, b]$ to obtain

$$\frac{E(T_i)}{T_i^*} = T_i^* E\left(\frac{1}{T_i}\right) \leqslant 1.043.$$

Case (b). T_i^* lies in $(2^{p_i}b, 2^{p_i+1}a]$, i.e., $T_i^* = 2^{p_i}w_i = 3.2^{p_i}z_i$, where $w_i \in (b, 2a]$ and $z_i \in [1, 2]$.

Then

$$\begin{aligned} E\left(pY_i^1 + qT_i^2\right) &= p\left(2^{p_i+2}\right) + q\left(3.2^{p_i}P(3z_i < Y)\right. \\ &\qquad \left. + 3.2^{p_i+1}P(3z_i \geqslant Y)\right) \\ &= T_i^*\left(p\frac{4}{w_i} + q\frac{3\left(1+F'(w_i)\right)}{w_i}\right), \end{aligned}$$

and

$$E\left(p\frac{1}{T_i^1} + q\frac{1}{T_i^2}\right) = \frac{1}{T_i^*}\left(p\frac{w_i}{4} + q\left(1 - \frac{F'(w_i)}{2}\right)\frac{w_i}{3}\right).$$

We have chosen $F'(\cdot)$ such that

$$p\frac{4}{w_i} + q\frac{3\left(1+F'(w_i)\right)}{w_i} = p\frac{w_i}{4} + q\left(1 - \frac{F'(w_i)}{2}\right)\frac{w_i}{3}.$$

With this choice of F', again we have

$$\begin{aligned} \frac{E\left(pT_i^1 + qT_i^2\right)}{T_i^*} &= T_i^*\left(p\frac{1}{T_i^1} + q\frac{1}{T_i^2}\right) \\ &\leqslant \max_{w_i \in (b, 2a]}\left(p\frac{4}{w_i} + q\frac{3\left(1+F'(w_i)\right)}{w_i}\right) \\ &\leqslant 1.043. \end{aligned}$$

Hence the result follows. □

5. CONCLUDING REMARKS

In this paper, we proposed a new randomized rounding approach to several multistage inventory/distribution lot sizing problems. The approach simplifies and extends the proofs to several well-known results in this area, especially in the case of constrained lot sizing problems. More importantly, through the use of a stronger formulation we obtain an improved approximation bound for the joint replenishment problem under the fixed base planning period model. Our result shows that the class of stationary policies is within 95.8% of the possibly dynamic optimal policies under this model.

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