

Staggering Periodic Replenishment in Multivendor JIT Environments

Sin-Hoon Hum

Department of Decision Sciences, NUS Business School, National University of Singapore,
Singapore 117591, bizhumsh@nus.edu.sg

Moosa Sharafali

Department of Management, Faculty of Economics and Commerce, University of Melbourne,
Melbourne, Victoria 3010, Australia, moosa@unimelb.edu.au

Chung-Piaw Teo

SKK Graduate School of Business, Sungkyunkwan University, Seoul, South Korea 110745, teocp@skku.edu, and
Department of Decision Sciences, NUS Business School, National University of Singapore, Singapore 117591, bizteocp@nus.edu.sg

The delivery scheduling problem studied in this paper was motivated by the operation in a large personal computer assembly plant, which was using multisourcing for some of its materials. The company's objective was to design a delivery schedule so that the average inventory level in the factory was minimized. We show that the problem is intimately related to a classical inventory staggering problem, where the focus is on the computation of the peak inventory level associated with the replenishment policy. This connection allows us to show that the delivery scheduling problem is NP-hard. For the two-vendor case with *integral* replenishment intervals, we propose a generalized form of Homer's scheduling heuristic and obtain performance bounds for the classical inventory staggering problem. Our analysis uses the Chinese remainder theorem in an interesting way. The approach can be generalized to the case with more than two vendors, leading to a strong linear-programming-based lower bound for the inventory staggering problem. We illustrate this technique for the case in which all the replenishment intervals are relatively prime, establishing a bound that is not greater than 140% of the optimal. We examine the implications of these results to the delivery scheduling problem.

Subject classifications: logistics; inventory management; multisourcing; Chinese remainder theorem.

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1. Introduction

A major focus of the just-in-time (JIT) manufacturing system is to improve product quality and productivity through the elimination of waste from all operations. Waste can be eliminated by facilitating frequent shipment of purchased parts in small lots and by manufacturing small lots frequently. Small lot sizes contribute to higher productivity in a firm through lower levels of inventory and scrap, high product quality, and increased flexibility. Thus, a prime prerequisite for a successful JIT system is the effective linkage of the JIT producer's purchasing department and the vendor's marketing department.

Our study is motivated by precisely these issues in such JIT inventory systems and is based on a large personal computer assembly operation. In the plant that we studied, an internal JIT material flow control system within its production lines has been in place for some time. Under this existing JIT system, however, materials from external vendors are scheduled to arrive in periodic cycles. Broad parameters on the *frequency* and *size* of each delivery are pre-negotiated in the supply contracts. The details regarding *delivery frequency and timings* depend largely

on the distance between the vendors and the factory. They depend on the replenishment interval, i.e., the time it takes to bring the order from the factory back to the vendor; the time it takes to fill the order; and the time it takes to transport the filled order back to the JIT factory. They also depend on the number of trucks each vendor has committed in servicing the JIT factory. Details regarding *delivery quantities* are initially agreed upon one week in advance, based on the upcoming production plan.

These issues of delivery timings and quantities are complicated by the fact that the company uses more than one vendor for certain parts to minimize the risk of shortages of parts. The volume of business for each vendor is pre-determined during the broad contract negotiation process and thus is an input to the problem. As far as possible, the vendors would like a stable flow of parts into the factory, on a periodic basis. This moves the system a step closer to the ideal of having the vendor and the customer closely coupled. When there are multiple vendors (each with multiple trucks), the schedules of the vendors (trucks) are usually time-spliced so that each vendor can potentially supply out of his production line, running a continuous operation with steady volume.

The system described is nonetheless a push system because material will flow from vendor to the factory according to the given schedule, regardless of production rate changes at the factory. Expeditors are hired to manage the dynamic deviations between delivery and production requirement. This is also because some vendors have difficulty adhering to the delivery timings and quantities. Continuous monitoring by the expeditors is necessary to ensure that the production operation has material to run on. As an additional safeguard, a large safety stock is kept.

As a first step toward a JIT linkage with their vendors, management felt that a kanban control system can be put in place to tie vendors to the actual production rate in the factory. Taking the delivery schedule as given, controlling the number of kanbans to be injected into the vendor-factory interface, and monitoring the allocation of kanbans among the vendors, the latter's detailed delivery quantities can be modified to reflect the actual rates of production pull from the factory. Our initial study was therefore of helping management in this *operational-level* problem of designing and implementing a kanban system with their vendors, taking the existing delivery schedules as given.

Although the frequencies and delivery quantities from the vendors allow the factory to meet its hourly production demand, it became obvious to us that the timings of the vendors' deliveries had an adverse effect on the average inventory level in the factory. To minimize the average inventory level, an obvious strategy is for the vendors to deliver only when the inventory level of the part at the factory drops to zero. This is easy to coordinate if the vendors can expedite deliveries on request, but it will negate the effectiveness of the kanban system being put in place. Furthermore, this will cause severe disruption to the smooth flow of parts between the vendors and the factory.

Another way to reduce the average inventory level is to coordinate the flow of materials among the different vendors while maintaining the periodicity of the vendors' delivery schedule. This can be achieved by staggering the delivery timings of the trucks from the different vendors. This is clearly a higher, more *strategic-level* problem than the initial operational problem of kanban system design and implementation.

In this paper, we study this strategic level problem (henceforth referred to as the *delivery scheduling* problem) of finding the best schedule for the deliveries of the supplies from the vendors. Our methodology is to first establish a relationship between this problem and the well-studied *inventory staggering* problem, and then to exploit this relationship to obtain a good schedule for the *delivery scheduling* problem.

Now, the *inventory staggering* problem usually appears in the context of management of the single-resource constrained multi-item inventory system (in short, SRMIS), studied in Anily (1991), Gallego et al. (1992, 1996), and Hariga and Jackson (1996). The SRMIS problem seeks to determine the optimal order quantities and replenishment epochs for each item so as to minimize average set-up

and inventory-related costs over all item-periodic policies, given by

$$\sum_{i=1}^M \left(\frac{K_i}{T_i} + H_i T_i \right) + \lambda S^{\max}(\mathcal{P}^*(T_1, T_2, \dots, T_M)),$$

where K_i and H_i are constants determined by the ordering and inventory holding costs of the i th item, λ is the cost of unit storage requirement, and $\mathcal{P}^*(T_1, T_2, \dots, T_M)$ denotes an optimal inventory staggering policy in the class of optimal policies (see Gallego et al. 1996 and Hariga and Jackson 1996 for details on the class of optimal policies). The peak storage requirement, denoted by $S^{\max}(\mathcal{P}^*(T_1, T_2, \dots, T_M))$, depends on the order intervals T_i and on how the orderings are staggered relative to one another.

This model was first proposed and studied by Homer (1966), who assumed that all items share one common order interval and derived the optimal solution. Later, Page and Paul (1976), Zoller (1977), and Hall (1988) independently rediscovered Homer's result and proposed heuristics for cases with more than one possible order intervals. The heuristics first partition the items into clusters, with all items in each cluster sharing a common order interval, thus enabling optimal staggering according to Homer's solution. However, no attempt was made to deal with the interactions among different clusters. Goyal (1978) argued that if different clusters are arranged properly, further reduction in warehouse space requirement is possible. This gives rise to the *inventory staggering* problem, in which the objective is to stagger the given clusters to minimize the peak inventory storage requirement. The optimal solution for the two-item inventory staggering problem was obtained by Hartley and Thomas (1982) and Thomas and Hartley (1983). Restricting herself to the class of policies called the stationarity-between-orders policies (SOSI policies), Anily (1991) performed a worst-case analysis for a class of partition heuristic. She proved a lower bound on the minimum required warehouse space and on the total cost for this class of policies. Gallego et al. (1996) showed that Anily's lower bound on the minimum required warehouse size for SOSI policies is in fact a lower bound for any feasible policy. They constructed a $\sqrt{2}$ -approximation algorithm by exploiting the observation that the peak inventory level of *any* staggering policy is within two times the minimum. The $\sqrt{2}$ approximation bound for the SRMIS problem can be improved only if we can manage to address the impact of staggering on the peak inventory storage requirement. Teo et al. (1998) studied a special case of the staggering problem. When the intervals T_i s are nested (i.e., T_i divides T_j whenever $i < j$), they obtained an improved lower bound on the peak usage of warehouse space for the inventory staggering problem. This gave rise to a 15/8 approximation algorithm for this problem. In the case $M = 2$, they obtained a staggering policy with a performance bound

of 4/3. Note that their results apply only if the order intervals satisfy the nested property.

The above staggering problem is also of interest in logistics management. For instance, Pesenti and Ukovich (2000) illustrated the relevance of this problem in replenishment planning to reduce holding cost and space utilization. Let h_k denote the headway (i.e., gap) between the k th and $k + 1$ th order. Pesenti and Ukovich seek to derive a staggering policy to minimize the maximum headways (i.e., $\min \max_k h_k$) and show that this problem has relevance in a variety of settings. For instance, these problems are relevant in the case pointed out by Hall (1991), where a product can be supplied to the same customer with different frequencies. Unfortunately, due to its complexity, very few results have been obtained for this problem. Burkard (1986) proved some fundamental results in the case of two periodic orders and obtained the optimal staggering policies in this special case. Pesenti and Ukovich extended his insights to a more general version of the problem.

Review of the literature on multisourcing and its benefits can be found in Kratz and Cox (1982), Greer and Liao (1986), and Horowitz (1986). Note that our focus here is simply on the delivery-coordination aspect of multisourcing.

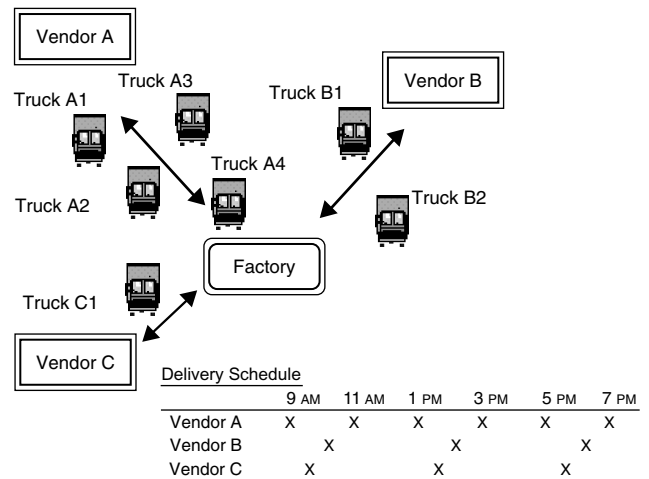
In the next section, we formally define this problem and the associated *inventory staggering* problem together with notation to be used. In §3, we demonstrate that the delivery scheduling problem and the inventory staggering problem are essentially equivalent. This relationship leads us to the fact that the delivery scheduling problem is NP-hard. For the single-vendor case, we provide an optimal schedule here. The dual-sourcing case is taken up in §4.2. For *integral* replenishment intervals, we obtain an approximation algorithm with a performance bound of at most 134% of the optimal. In §4.3, we treat the case of more than two vendors who all have relatively prime replenishment intervals. We again obtain an approximation algorithm whose performance bound is not more than 140% of the optimal. In proving the performance bounds, we use the Chinese remainder theorem in an interesting way. In §5, we examine the implications of these results for the delivery scheduling problem. In the final section, we conclude the paper with a brief review on the results obtained and some possible extensions.

2. Delivery Scheduling and Inventory Staggering

2.1. Delivery Scheduling Problem: Definition

As noted above, the delivery scheduling problem is to find the optimal schedule for the multiple trucks used in delivering materials from multiple vendors. With the replenishment times being different for different vendors, the objective is to minimize the average inventory level in the factory, subject to the constraints that vendors deliver in a periodic pattern and that there is no stock out in the factory.

Figure 1. Three vendors supplying a factory with truck fleets.



To make the matters clear, we illustrate the delivery scheduling problem with an example that would also be used to define the parameters and the decision variables of the problem.

Illustration of the Delivery Scheduling Problem.

Figure 1 shows an example of a material delivery system. In the example, the factory has three vendors, A, B, and C.

- Vendor A has four trucks making the delivery. Each circuit is completed in eight hours. Spacing the trucks equally, this vendor makes a delivery at the factory every two hours, at 0900, 1100, 1300, etc.

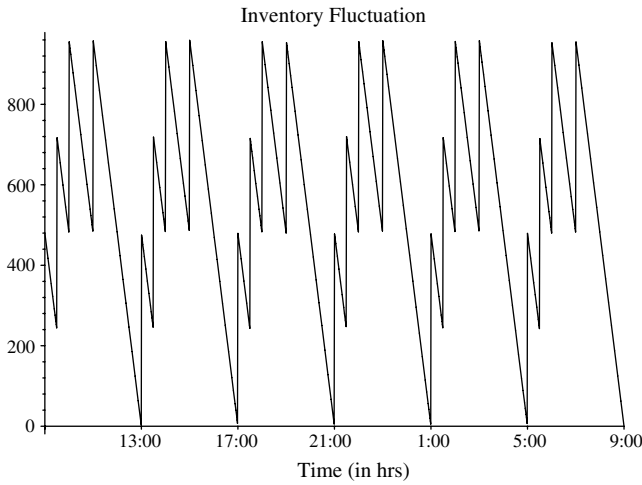
- Vendor B has two trucks making the delivery. Each circuit is completed in eight hours. Spacing the trucks equally, the vendor makes a delivery at the factory every four hours, at 1000, 1400, etc.

- Similarly, vendor C makes deliveries every four hours with just one truck, at 0930, 1330, etc.

Each truck, after making its delivery, picks up a new order. This new order quantity is the quantity the same truck is required to deliver on its next trip. In this way, vendor C in Figure 1 is different from A and B. The truck C1 that carries back the last order is also the one that makes the next delivery. The interdelivery and order lead times are four hours each. For vendor A, though the interdelivery time is two hours, the order lead time is eight hours. Vendor B, with the same interdelivery time as vendor C, has a larger order lead time of eight hours. Note that the above delivery schedule is most natural for the vendors, as they tend to space out their deliveries to the factory equally over the time horizon.

Suppose that the factory utilizes the parts at a constant rate of 480 units in every hour. Assume further that each delivery from the vendors brings 480 units into the factory. For vendor A, with an interdelivery time of two hours, this supply is equivalent to satisfying 240 units per hour of the demand at the factory. Similarly, vendor B delivers

Figure 2. Inventory level fluctuation.



120 units per hour, and vendor C delivers 120 units per hour. Thus, on a per-hour basis, the supply is enough to meet the demand for parts in the factory.

Figure 2 shows the fluctuation in the inventory level at the factory, according to the delivery schedule as shown in Figure 1. The average inventory level at the factory over an eight-hour period (after which the schedule repeats itself) is 540 units. The *maximum* inventory level reached is 960 units.

It should be noted that although the delivery times are periodic, the sequence of deliveries can start at any point in time. One can easily construct other schedules for the deliveries, which would result in different values for the average and the peak inventory levels in the factory.

Specifically in the sequel, we consider a factory that is supported by M vendors. Without loss of generality, we assume that the part is consumed at a rate of 1 unit per unit time. Vendor i has n_i trucks where $\sum_{i=1}^M n_i = n$, with each truck making a delivery to the factory at every T_i unit interval. We assume that vendors with the same replenishment intervals are grouped together, which allows us to treat these as deliveries from a single vendor. Thus, the trucks from this vendor (or from this group of vendors) may have different delivery quantities. With this grouping of vendors, we can further assume that the replenishment intervals for the vendor groups are all distinct. In what follows, we will take each vendor group to be a single vendor. Note that without loss of generality, we may assume that all replenishment intervals are rational. Otherwise, if the ratio of the replenishment intervals of two vendors is irrational, then no matter how best we try to coordinate the delivery schedule, because of the periodicity of the delivery timings we cannot avoid deliveries from the two vendors to the factory occurring at approximately the same time, some time into the future. Henceforth, we assume that the replenishment intervals are rational, and thus by scaling, we can assume that the intervals are all integers.

The unit demand rate in the factory is divided among the vendors in the proportion $\Gamma_1 : \Gamma_2 : \dots : \Gamma_M$, with $\sum_{i=1}^M \Gamma_i = 1$. As pointed out earlier, this allocation is decided at the time of contract negotiation with the vendors. Each vendor, in turn, divides this allocation among the trucks the vendor possesses. If truck j belongs to vendor i , we let γ_{ij} denote the proportion of demand allocated to that truck. Note that $\sum_{j=1}^{n_i} \gamma_{ij} = \Gamma_i$. As the delivery interval from vendor i is T_i , in each trip, truck j will bring $\gamma_{ij}T_i$ unit of inventory to the factory.

For the example described above, we have $M = 3$ with $n_1 = 4, n_2 = 2, n_3 = 1$, and $n = 7$. Further, $T_1 = 8, T_2 = 8$, and $T_3 = 4$. The demand allocation is $\Gamma_1 : \Gamma_2 : \Gamma_3 = 1/2 : 1/4 : 1/4$. For each i , γ_{ij} are equal.

The decision here is to obtain a delivery schedule, so the decision variables are

t_{ij} : instant in $(0, T_i]$ at which truck j from vendor i makes a delivery.

We are now in a position to formulate the delivery scheduling problem. Note that the sequence of deliveries from vendor i will take place at regular T_i interval. Truck j from vendor i brings an amount of $\gamma_{ij}T_i$ material to the factory, which will be consumed at a rate of γ_{ij} . Overall, the total consumption at the factory is thus $\sum_{ij} \gamma_{ij} = \sum_i \Gamma_i = 1$.

Let $g_{ij}(t)$ be the amount of inventory of parts brought into the factory by truck j from vendor i , at time t . To prevent stocking out at the factory, we need a certain amount of inventory in the factory at time 0. For example, we may assume that the inventory level at time 0 is $\sum_{i,j} \gamma_{ij}t_{ij}$, where $\gamma_{ij}t_{ij}$ units are attributed to truck j from vendor i . In this way, the inventory level of the parts brought into the factory, by truck j from vendor i , can be represented by the function

$$g_{ij}(t) = \begin{cases} \gamma_{ij}(t_{ij} - t), & 0 \leq t < t_{ij}, \\ \gamma_{ij}(T_i + t_{ij} - t), & t_{ij} \leq t < T_i + t_{ij}, \\ g_{ij}(t - T_i), & t \geq T_i + t_{ij}. \end{cases}$$

Note that $\sum_{ij} g_{ij}(0) = \sum_{ij} \gamma_{ij}t_{ij}$.

Using this amount of initial inventory, the factory will have no problem supporting the delivery schedule, given any t_{ij} . However, it is stocking more than the necessary material required. Let

$$S^{\min}(\mathcal{P}) = \min_{t \geq 0} \left(\sum_{i=1}^M \sum_{j=1}^{n_i} g_{ij}(t) \right),$$

where $S^{\min}(\mathcal{P})$ is called the *protection level* of the scheduling policy \mathcal{P} . At any point in time, there is at least $S^{\min}(\mathcal{P})$ units of inventory in the factory. By reducing this amount of inventory from the initial inventory at time 0, we will still have enough material to support the delivery schedule t_{ij} , without incurring stock-out. Let

$$G(t) = \left(\sum_{i=1}^M \sum_{j=1}^{n_i} g_{ij}(t) \right) - S^{\min}(\mathcal{P})$$

denote the new inventory level at the warehouse at time t , after reducing the initial inventory from $\sum_{ij} \gamma_{ij} t_{ij}$ to $\sum_{ij} \gamma_{ij} t_{ij} - S^{\min}(\mathcal{P})$ units. Note that $G(t) \geq 0$, and

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T G(t) dt$$

is the average inventory level associated with the delivery schedule given by t_{ij} , using the minimum level of initial inventory at time 0. The delivery scheduling problem can be stated as follows:

$$\min_{t_{ij}: T_i \geq t_{ij} > 0} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \left(\left(\sum_{i=1}^M \sum_{j=1}^{n_i} g_{ij}(t) \right) - S^{\min}(\mathcal{P}) \right) dt.$$

Note that the average inventory level for all items is given by

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \sum_{i=1}^M \sum_{j=1}^{n_i} g_{ij}(t) dt = \frac{1}{2} \sum_{i=1}^M \sum_{j=1}^{n_i} \gamma_{ij} T_i,$$

because the inventory level for parts brought in by truck j from vendor i fluctuates like an EOQ model, and has average inventory level of $(1/2)\gamma_{ij}T_i$. The average inventory level in this case *does not* depend on the staggering policy $\{t_{ij}\}$.

The challenge in the delivery scheduling problem is thus to *maximize* the protection level, $S^{\min}(\mathcal{P})$.

2.2. Inventory Staggering Problem: Definition

Consider a multi-item warehouse supplying *distinct* parts to an assembly plant. The primary issue in the management of this warehouse is to ensure that there is always enough space to accommodate all the deliveries with no disruption to the assembly operation. We assume that the management of each part in the warehouse follows an EOQ model. It is evident that the demand for the warehouse space is the highest when all the parts deliveries arrive at the same time. By staggering or time-phasing these deliveries, the peak demand on the warehouse space can be moderated (Hariga and Jackson 1996). Staggering orders not only results in efficient use of warehouse space, but also reduces the cost of holding parts in the warehouse.

For the above inventory staggering problem, consider a policy \mathcal{P} . Let n_i be the number of parts with reorder interval T_i , $i = 1, 2, \dots, M$. Let γ_{ij} denote the rate of consumption of part j with re-order interval T_i . Let this part be denoted as part (i, j) .

Let t_{ij} be the delivery instant for part (i, j) that takes place in $(0, T_i]$. After this, the delivery of that part will take place at a regular T_i interval. Each delivery brings an amount of $\gamma_{ij}T_i$ material to the plant, which will be consumed at a rate of γ_{ij} . As in the case of the delivery scheduling problem, let $g_{ij}(t)$ denote the amount of inventory of part (i, j) in the warehouse at time t . Note that the inventory level at time 0 is $\gamma_{ij}t_{ij}$ for part (i, j) , so that the

warehouse would have enough material for part (i, j) to last till the first delivery after time 0. Thus, as before,

$$g_{ij}(t) = \begin{cases} \gamma_{ij}(t_{ij} - t), & 0 \leq t < t_{ij}, \\ \gamma_{ij}(T_i + t_{ij} - t), & t_{ij} \leq t < T_i + t_{ij}, \\ g_{ij}(t - T_i), & t \geq T_i + t_{ij}. \end{cases}$$

The inventory staggering problem aims to minimize the peak inventory usage and is defined as

$$\min_{t_{ij}: T_i \geq t_{ij} > 0} \max_t \left(\sum_{i=1}^M \sum_{j=1}^{n_i} g_{ij}(t) \right).$$

We can define an analogous notion of the protection level, for any staggering policy, to be the minimum level of the inventory of all the parts over all time. This is again denoted by $S^{\min}(\mathcal{P}) = \min_{t \geq 0} (\sum_{i=1}^M \sum_{j=1}^{n_i} g_{ij}(t))$.

3. Average, Minimum, and Maximum Inventory Levels

Next, we show that the delivery scheduling problem (maximize minimum inventory level) is connected to the classical inventory staggering problem (minimize maximum inventory level). To this end, we first identify below a dual construction that yields a useful and interesting conservation law in this class of scheduling problems.

For any staggering schedule \mathcal{P} , let $S^{\max}(\mathcal{P})$ be the peak storage usage needed. Let $S^{\min}(\mathcal{P})$ be the minimum amount of inventory in the factory at all time (i.e., the protection level for the schedule \mathcal{P}). Although we define the staggering policy for time $t \geq 0$, it is convenient to extend the staggering policy to all t from $-\infty$ to ∞ , while maintaining the fixed ordering interval of T_i for each i .

For each schedule \mathcal{P} , we can construct a delivery schedule \mathcal{Q} (called dual of policy \mathcal{P}) in the following way:

- Let $\{t_{ij}\}$ be the staggering solution obtained under schedule \mathcal{P} in the staggering problem.
- Let $\{t'_{ij} \equiv -t_{ij} \pmod{T_i}\}$ be the staggering solution obtained for part (i, j) in \mathcal{Q} .

Note that the first delivery of part (i, j) in schedule \mathcal{P} after time 0 is at time t_{ij} , whereas the first delivery for schedule \mathcal{Q} after time 0 is at time $T_i - t_{ij}$. The last delivery before time 0 in schedule \mathcal{Q} is thus at time $-t_{ij}$. Intuitively, the delivery schedule \mathcal{Q} is obtained by reversing the direction of time in the staggering schedule \mathcal{P} .

THEOREM 1. For all staggering schedules \mathcal{P} , $S^{\max}(\mathcal{P}) + S^{\min}(\mathcal{Q}) = \sum_i \Gamma_i T_i$.

PROOF. For ease of exposition, we can assume that each vendor has only one truck, for if a vendor has more than one truck, then each truck can be imagined to be owned by a different vendor. Thus, the proof for the multiple trucks extension is straightforward. Hence, part (i, j) is now written simply as part i .

Let $L(\mathcal{P}, t)$ denote the aggregate inventory level in the factory at time t . Let $a_i(t)$ be the time from t to the next arrival of order for part i . Hence,

$$L(\mathcal{P}, t) = \sum_{i=1}^M \Gamma_i a_i(t).$$

Let $t - b_i(t)$ denote the instant of the most recent delivery for part i before time t in \mathcal{P} .

In the schedule \mathcal{Q} , this corresponds to the time from $-t$ to the next delivery for part i . This duration is $b_i(t)$ (by construction of \mathcal{Q}). Hence,

$$L(\mathcal{Q}, -t) = \sum_{i=1}^M \Gamma_i b_i(t).$$

Because

$$a_i(t) + b_i(t) = T_i \quad \text{for all } i,$$

we have

$$L(\mathcal{P}, t) + L(\mathcal{Q}, -t) = \sum_{i=1}^M \Gamma_i T_i \quad \text{for all time } t.$$

Hence, $L(\mathcal{P}, t)$ attains its maximum if and only if $L(\mathcal{Q}, -t)$ attains its minimum, so $S^{\max}(\mathcal{P}) + S^{\min}(\mathcal{Q}) = \sum_i \Gamma_i T_i$. \square

Note that by symmetry,

$$S^{\max}(\mathcal{Q}) + S^{\min}(\mathcal{P}) = \sum_i \Gamma_i T_i$$

and

$$S^{\max}(\mathcal{P}) - S^{\min}(\mathcal{P}) = S^{\max}(\mathcal{Q}) - S^{\min}(\mathcal{Q}).$$

If $S^{\max}(\mathcal{P}) \leq S^{\max}(\mathcal{Q})$, then $S^{\min}(\mathcal{P}) \leq S^{\min}(\mathcal{Q})$.

Let \mathcal{P}^* be the schedule that minimizes the peak inventory level in the factory. Let \mathcal{Q}^* be the dual schedule obtained by reversing the direction of time using \mathcal{P}^* . For any policy \mathcal{P} with associated dual schedule \mathcal{Q} ,

$$S^{\max}(\mathcal{P}^*) \leq S^{\max}(\mathcal{P}), \tag{1}$$

and therefore,

$$S^{\min}(\mathcal{Q}^*) \geq S^{\min}(\mathcal{Q}). \tag{2}$$

The above result shows that the optimal schedule that maximizes the protection level can be obtained from the dual of the optimal policy that minimizes the peak storage usage. Because the delivery scheduling problem is equivalent to the protection level maximization problem, this shows that the delivery scheduling problem is as hard as the inventory staggering problem.

Gallego et al. (1992) showed that the inventory staggering problem is NP-hard (cf. Theorem 1 in Gallego et al. 1992). In fact, their proof actually shows that the staggering problem is NP-complete in the strong sense, even if only one item has a different re-order interval. Combining this with Theorem 1, we have Corollary 1.

COROLLARY 1. *The delivery scheduling problem is strongly NP-hard, even in the case when all replenishment intervals are in $\{1, k\}$, where $k > 1$ is some integer.*

A well-known result for the inventory staggering problem states that (cf. Gallego et al. 1996)

$$S^{\max}(\mathcal{P}) \geq \frac{1}{2} \sum_{i,j} \gamma_{ij} (1 + \gamma_{ij}) T_i \tag{3}$$

for all inventory staggering policy \mathcal{P} . We have, via Theorem 1, the following lower bound for the delivery scheduling problem.

THEOREM 2. *The minimum average inventory level in the delivery scheduling problem*

$$\min_{t_{ij} \geq 0, t_{ij} \leq T_i} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (G(t)) dt$$

is bounded below by

$$\frac{1}{2} \sum_{i,j} \gamma_{ij}^2 T_i.$$

PROOF. Note that

$$\min_{t_{ij} \geq 0, t_{ij} \leq T_i} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T G(t) dt = \frac{1}{2} \sum_{i,j} \gamma_{ij} T_i - \max_{\mathcal{Q}} S^{\min}(\mathcal{Q}).$$

From Theorem 1, $S^{\max}(\mathcal{P}) + S^{\min}(\mathcal{Q}) = \sum_{i,j} \gamma_{ij} T_i$, and from (3), we have

$$\begin{aligned} \min_{t_{ij} \geq 0, t_{ij} \leq T_i} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T G(t) dt &= \min_{\mathcal{P}} S^{\max}(\mathcal{P}) - \frac{1}{2} \sum_{i,j} \gamma_{ij} T_i \\ &\geq \frac{1}{2} \sum_{i,j} \gamma_{ij}^2 T_i. \quad \square \end{aligned}$$

4. Inventory Staggering Problem

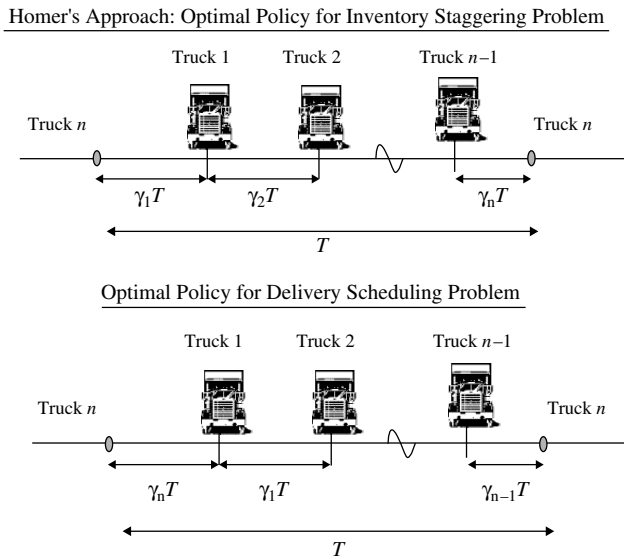
Because for certain choices of replenishment intervals the optimal protection level for the delivery scheduling problem can be arbitrarily small, it seems difficult to analyze the performance of heuristics for this class of problems. In the rest of this paper, we will circumvent this difficulty by focusing on the inventory staggering problem by minimizing the peak of the inventory level in the system. Our analysis focuses on the worst-case bound for the inventory staggering problem. Note that a good solution for the inventory staggering problem can be used to construct a good dual policy for the delivery scheduling problem.

4.1. Single-Vendor Case

$$(M = 1, n_1 = n, \gamma_{ij} = \gamma_j, T_i = T)$$

In the case of the single-vendor problem, it turns out that the inventory staggering problem (and hence the delivery scheduling problem) can be solved to optimality by a simple algorithm due to Homer (1966). This can be used

Figure 3. Optimal schedule for the delivery scheduling and inventory staggering problem.



to construct the optimal policy for the delivery scheduling problem: Arrange the trucks in any order, say from 1, 2, . . . , n, and stagger the deliveries of the trucks such that the intervals between truck j and $j + 1$ is $\gamma_j T$. This policy is precisely the dual of the optimal policy as constructed by Homer (1966). Note that the T_i s for all the trucks are identical, and $\sum_i \gamma_i = 1$ by definition. Figure 3 shows the optimal delivery schedule for both inventory staggering and delivery scheduling problems.

Homer’s policy for the inventory staggering problem works by observing that when truck 1 with a load of $\gamma_1 T$ arrives, the inventory in the factory would have depleted by an amount equivalent to $\gamma_1 T$ (the demand rate is 1, and the time between arrivals of truck n and truck 1 is $\gamma_1 T$). Hence, the load delivered by truck 1 is just enough to replace what has been consumed. This property holds for all other deliveries.

The optimal policy for the delivery scheduling works in the opposite manner: Truck 1 arrives only at the instant when the load brought by truck n has just been consumed in the factory; i.e., truck 1 arrives at a point when the inventory level in the factory drops to zero. This property extends to all other deliveries in the schedule. Hence, the optimal policy is similar to the JIT zero-inventory policy often practiced in the industry.

In the optimal solution to the delivery scheduling problem, the average inventory level attains a value $(1/2) \sum_{j=1}^n \gamma_j^2 T$. Note that this is the lower bound obtained in Theorem 2. The optimal peak inventory level in the corresponding inventory staggering problem attains a value $(1/2) \sum_{j=1}^n (\gamma_j + \gamma_j^2) T$.

Note that an implicit assumption in the above is that $\sum_i \gamma_i = 1$, the demand rate, so that the supply matches the demand. If $\sum_i \gamma_i = d$ for some demand rate d , then the

optimal schedule will have to be scaled by a factor d . The peak inventory level for the inventory staggering problem, in this case, will attain a value

$$\frac{1}{2} \sum_i (\gamma_i) \left(1 + \frac{\gamma_i}{d}\right) T.$$

In the rest of this section, we discuss the two-vendor and multivendors problem.

4.2. Two-Vendor ($M = 2$) Problem with General Ordering Intervals

We show that a simple heuristic can be used to construct a good staggering schedule for the two-vendor case, such that the peak inventory level will not be far off the optimal solution. Without loss of generality, we can assume that the delivery intervals of the two vendors are $T_1 = p$ and $T_2 = q$, where p and q are relatively prime and $p < q$. Otherwise, if T_1 and T_2 share a common factor, by proper scaling we can reduce the problem to this case where the intervals are relatively prime. Teo et al. (1998) established a $4/3$ approximation bound for the inventory staggering problem using a complicated policy for the case when the two intervals are nested (i.e., $p = 1$). We extend this result to the case in which the intervals are not necessarily nested. To aid our analysis, we use the Chinese remainder theorem, the statement of which follows.

THEOREM 3 (CHINESE REMAINDER THEOREM). *Let m_1, m_2, \dots, m_k be pairwise relatively prime integers. If a_1, a_2, \dots, a_k are any integers, then*

- *there exists an integer a such that $a \equiv a_i \pmod{m_i}, i = 1, 2, \dots, k$, and*
- *If $b \equiv a_i \pmod{m_i}, i = 1, 2, \dots, k$, then*

$$b \equiv a \pmod{m_1 m_2 \dots m_k}.$$

Now, let \mathcal{P}^* be the optimal policy for the two-vendor problem above. We split the schedule obtained into two parts, one for each vendor. Let U_i denote the instances where the inventory level due to vendor i alone (in schedule \mathcal{P}^*) is at its peak. Let S_i denote the corresponding peak inventory level for vendor i . By the periodic nature of the deliveries, the peaks will be reached at regular T_i unit intervals. Without loss of generality, we may also assume that $0 \in U_2$.

Let

$$d(U_1, U_2) = \min\{|u - v|: u \in U_1, v \in U_2\}.$$

We argue next that the distance $d(U_1, U_2)$ between the two sets U_1 and U_2 is small.

Let $u \in U_1$. Suppose that u lies between the integers N and $N + 1$, with $u = N + \alpha$. As an immediate consequence of the Chinese remainder theorem, there exists an integral solution to the equations

$$T \equiv 0 \pmod{q}, \quad T \equiv N \pmod{p}.$$

This assures that in the optimal policy, there exists a point T in time so that the inventory due to vendor 2 will be at its peak (because $0 \in U_2, T \equiv 0 \pmod{q}$), whereas the peak inventory level due to vendor 1 will be reached α time unit later.

Because the inventory for vendor i is consumed at a rate of $\sum_{k=1}^{n_i} \gamma_{i,k}$, the peak cumulative inventory level for \mathcal{P}^* is at least the inventory level at time $T + \alpha$:

$$S_1 + S_2 - \alpha \sum_{k=1}^{n_2} \gamma_{2,k} = S_1 + S_2 - \alpha \Gamma_2.$$

By the Chinese remainder theorem again, there exists an integral solution to the equations

$$T' \equiv 0 \pmod{q}, \quad T' \equiv (N + 1) \pmod{p}.$$

This assures that in the optimal policy, there exists a point T' in time so that the inventory due to vendor 2 will be at its peak, whereas the inventory due to vendor 1 would have peaked $(1 - \alpha)$ time unit earlier.

The inventory level at time T' (when vendor 2 attains peak inventory level) is another lower bound to the peak inventory level. This gives rise to another inequality:

$$S^{\max}(\mathcal{P}^*) \geq S_1 + S_2 - (1 - \alpha)\Gamma_1.$$

Note that

$$\min(\alpha\Gamma_2, (1 - \alpha)\Gamma_1) \leq \frac{\alpha}{q} \left(\sum_{k=1}^{n_2} \gamma_{2,k} q \right)$$

and

$$\min(\alpha\Gamma_2, (1 - \alpha)\Gamma_1) \leq \frac{1 - \alpha}{p} \left(\sum_{k=1}^{n_1} \gamma_{1,k} p \right).$$

Hence,

$$S^{\max}(\mathcal{P}^*) \geq S_1 + S_2 - \frac{1}{q/\alpha + p/(1 - \alpha)} \left(\sum_{i,k} \gamma_{i,k} T_i \right). \quad (4)$$

Because $S^{\max}(\mathcal{P}^*) \geq (1/2) \sum_{i,k} \gamma_{i,k} T_i$, we have

$$S_1 + S_2 \leq \left(1 + \frac{2}{q/\alpha + p/(1 - \alpha)} \right) S^{\max}(\mathcal{P}^*).$$

Note that the value S_i is obtained by examining the peak inventory storage of vendor i alone. It cannot be less than the value we would obtain if we stagger the deliveries from vendor i in intervals of $(\gamma_{i,k}/\Gamma_i)T_i$, using Homer's approach.

Now consider the following scheduling policy.

Generalized Homer's Policy: For $i = 1, 2, \dots$,

- Let L_i be the set of trucks from vendor i .
- Schedule the deliveries of the trucks from vendor i using Homer's approach for a single vendor.

Note that in the above scheduling policy, we do not attempt to coordinate the deliveries from *different* vendors. Furthermore, it is the most natural policy in use because vendors tend to space out their delivery intervals evenly, if the load assigned to each delivery is identical. The resulting policy which ignores the effect of staggering across two different vendors will attain a peak inventory level of at most $S_1 + S_2$. The worst-case performance of this policy will thus depend on the value

$$\left(1 + \frac{2}{q/\alpha + p/(1 - \alpha)} \right).$$

This bound is maximized when

$$1 - \alpha = \frac{\sqrt{pq} - p}{q - p},$$

with the corresponding worst-case value

$$1 + \frac{2}{(\sqrt{p} + \sqrt{q})^2}.$$

We tabulate the worst-case bounds for a few values of p, q :

p	q	Bound
2	1	1.343145751
3	1	1.267949192
3	2	1.202041029
4	3	1.143593539
5	2	1.150098818
5	3	1.127016654
5	4	1.111456180
6	5	1.091097700

For all values of p and q , note that we obtain an approximation algorithm of at most 1.343 for the two vendor situation.

It is interesting to note from the above analysis that if p and q are large, then as long as we ensure that the deliveries from each vendor are staggered using Homer's approach, the resulting staggering schedule will not be much worse off than the optimal policy. However, the better approximation bound for larger p and q arises mainly because coordination across different replenishment intervals is futile in these cases, it is thus easier to construct an algorithm with a better approximation bound. However, it does not imply that the factory is better off having two vendors with large relatively prime replenishment intervals p and q ; it simply means that the generalized Homer's policy is already very effective in these cases, and that there is no further need to synchronize the schedules for vendors with different replenishment intervals. The slightly larger worst-case bound, for say the case $p = 2, q = 1$, means that there is room for further improvement should we try to synchronize the schedule across different vendors. We refer the readers to Teo et al. (1998) for synchronization strategies across items with nested replenishment intervals.

We state the above result formally, in the most general case, as the following theorem.

THEOREM 4. For a two-vendor inventory staggering problem, with delivery intervals of T_1 and T_2 , as long as the schedule for each vendor follows Homer's policy, the peak inventory level of the resulting inventory staggering schedule is at most

$$\max_{0 \leq \alpha \leq 1} \left(1 + \frac{2 \gcd(T_1, T_2)}{T_2/\alpha + T_1/(1-\alpha)} \right) = 1 + \frac{2}{\left(\sqrt{T_2/(\gcd(T_1, T_2))} + \sqrt{T_1/(\gcd(T_1, T_2))} \right)^2}$$

times of the optimal.

It should be noted that the approximation bound is sensitive to the choice of the replenishment intervals T_i . It goes to show that the selection of T_i can play a big role in the effectiveness of the delivery scheduling policies. To illustrate this point, it is worthwhile to make a comparison of the peak inventory level attained when (i) $T_1 = 120$, $T_2 = 60$, and when (ii) $T_1 = 120$, $T_2 = 59$. To simplify the discussion, we assume that there are only two items each with replenishment interval T_1 and T_2 , and the consumption rate for the two items are identical (say 1/2 per unit time). In case (i), the generalized Homer's heuristic attains a worst-case bound of 1.343. This should not be too surprising, because the generalized Homer's heuristic does not attempt to synchronize the deliveries for the two items at all. However, if we stagger the deliveries so they are at time periods $\{\dots, 0, 120, 240, \dots\}$ and $\{\dots, 30, 90, 150, \dots\}$, then it is easy to see that the peak inventory level is attained at time 0 with inventory level of only 75 units. On the other hand, in case (ii), the generalized Homer's heuristic attains a worst-case bound of 1.00576. However, because the replenishment intervals are relatively prime, there will be an instance in time where the two items will be delivered to the factory at around the same time, raising the inventory level to close to $60 + 29.5 = 89.5$ units. This is higher than what we can achieve with better synchronization in the first case. It is thus obvious that case (i) will be preferred to case (ii), although the approximation bound in case (i) is better than the bound obtained in case (ii). In general, finding the best combination for T_1 and T_2 seems to be an exceedingly difficult problem.

In the rest of this section, we describe how the analysis can be extended to the case with more than two vendors. In particular, we show that the generalized Homer's policy is a 1.4-approximation algorithm, when all replenishment intervals are relatively prime.

4.3. Multivendor Case

Let \mathcal{P}^* be an optimal policy for the inventory staggering problem with M vendors. We split the schedule obtained into M parts, one for each vendor. Let U_i denote the instances where the inventory level due to vendor i alone (in schedule \mathcal{P}^*) is at its peak. Let S_i denote the corresponding peak inventory level. By the periodic nature of

the deliveries, the peaks will be reached at regular T_i unit intervals. Without loss of generality, we assume that $0 \in U_1$.

Consider the case where the delivery intervals T_i are all relatively prime.

Let $u_i \in U_i$, $i > 1$. Suppose that u_i lies between the integer N_i and $N_i + 1$, with $u_i = N_i + \alpha_i$. We order the α_i , $i = 1, 2, \dots, M$ such that $\alpha_1 = 0 \leq \alpha_2 \leq \alpha_3 \leq \dots \leq \alpha_M$.

As an immediate consequence of the Chinese remainder theorem, there exists an integral solution to the equations

$$T \equiv 0 \pmod{T_1}, \quad T \equiv N_i \pmod{T_i} \quad \text{for all } i > 1.$$

This assures that in the optimal policy, there exists a point T in time so that the inventory due to vendor 1 will be at its peak, whereas the peak inventory level due to vendor i will be reached α_i time unit later.

Consider the total inventory at time $T + \alpha_M$. The inventory level due to vendor M is at its peak, and the inventory level due to vendor j ($j \neq M$) is at most $(\alpha_M - \alpha_j)\Gamma_j$ off its peak, because the peak was attained only $\alpha_M - \alpha_j$ time units ago and the maximum inventory depletion rate is Γ_j . Thus, we obtain a lower bound for the optimal total inventory to be

$$S^{\max}(\mathcal{P}^*) \geq \sum_{j=1}^M S_j - \sum_{j=1}^M \alpha_{Mj} \Gamma_j, \quad (5)$$

where

$$\alpha_{Mj} = \alpha_M - \alpha_j, \quad j = 1, 2, \dots, M. \quad (6)$$

Similarly, for any given $i = 2, \dots, M$, by the Chinese remainder theorem again, there exists an integral solution to the equations

$$T \equiv 0 \pmod{T_1},$$

$$T \equiv N_j \pmod{T_j} \quad \text{if } j < i, \quad j \neq 1,$$

$$T \equiv (N_j + 1) \pmod{T_j} \quad \text{if } j \geq i.$$

This assures that in the optimal policy, there exists a time T at which the inventory due to vendor 1 will be at its peak, whereas the peak inventory level due to vendor j ($j < i$, $j \neq 1$) would be reached α_j time units later. For all vendors j with $j \geq i$, the inventory would have peaked $1 - \alpha_j$ time units earlier. Now, as before, consider the total inventory at time $T + \alpha_i$. The inventory due to vendor j ($j < i$) would have reached its peak $\alpha_i - \alpha_j$ units earlier, whereas the inventory due to vendor j ($j \geq i$) would have reached its peak $\alpha_i + (1 - \alpha_j)$ units earlier. We obtain other lower bounds for the optimal total inventory to be

$$S^{\max}(\mathcal{P}^*) \geq \sum_{j=1}^M S_j - \sum_{j=1}^M \alpha_{ij} \Gamma_j, \quad i = 2, 3, \dots, M, \quad (7)$$

where

$$\alpha_{ij} = \begin{cases} \alpha_i - \alpha_j & \text{if } j < i, \\ 0 & \text{if } j = i, \\ \alpha_i + 1 - \alpha_j & \text{if } j > i. \end{cases}$$

Note that $\alpha_{ij} + \alpha_{ji} = 1$ for all $i \neq j$.

Let

$$z = \min_{1 \leq i \leq M} \left\{ \sum_{j=1}^M \alpha_{ij} \Gamma_j \right\}. \quad (8)$$

Let δ be such that

$$z = \frac{\delta}{2} \sum_{j=1}^M \Gamma_j T_j. \quad (9)$$

Using the fact that

$$S^{\max}(\mathcal{P}^*) \geq \frac{1}{2} \sum_{j=1}^M \Gamma_j T_j$$

and

$$S^{\max}(\mathcal{P}^*) \geq \sum_{j=1}^M S_j - z,$$

we obtain that

$$\sum_{j=1}^M S_j \leq (1 + \delta) S^{\max}(\mathcal{P}^*). \quad (10)$$

Let Z_H be the peak inventory storage obtained using the generalized Homer's policy. Clearly, $Z_H \leq \sum_j S_j$. Hence, the generalized Homer's policy is a $(1 + \delta)$ -approximation algorithm.

To find the worst-case behavior of the generalized Homer's policy, we need only to solve the following LP:

$$\begin{aligned} z^* = \max z, \\ \text{s.t. } \sum_{j=1}^M \alpha_{ij} \Gamma_j \geq z, \quad i = 1, 2, \dots, M, \\ \alpha_{ij} + \alpha_{ji} = 1 \quad \text{for all } i \neq j, \\ \alpha_{ii} = 0 \quad \text{for all } i = 1, 2, \dots, M, \\ \alpha_{ij} \geq 0 \quad \text{for all } i \neq j. \end{aligned}$$

By taking

$$\delta^* = \frac{2z^*}{\sum_{j=1}^M \Gamma_j T_j},$$

we obtain a worst-case performance ratio of $(1 + \delta^*)$.

In the remainder of the paper, we focus on finding a bound for δ^* without solving the LP. In fact, we note from aggregating constraints (11) that

$$\sum_{i=1}^M \Gamma_i \left(\sum_{j=1}^M \alpha_{ij} \Gamma_j \right) \geq \sum_{i=1}^M \Gamma_i z,$$

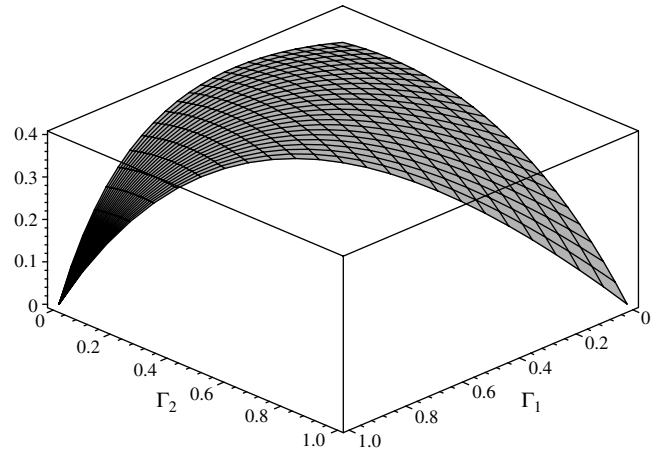
and because $\sum_{i=1}^M \Gamma_i = 1$, $\alpha_{ij} + \alpha_{ji} = 1$, and $\alpha_{ii} = 0$,

$$z^* \leq \sum_{i=1}^M \sum_{j>i} \Gamma_i \Gamma_j \quad (16)$$

$$= \frac{1}{2} \left(\left(\sum_{i=1}^M \Gamma_i \right)^2 - \sum_{i=1}^M \Gamma_i^2 \right) \quad (17)$$

$$= \frac{1}{2} \left(1 - \sum_{i=1}^M \Gamma_i^2 \right). \quad (18)$$

Figure 4. Plot of $g(\Gamma_1, \Gamma_2)$ with $\Gamma_1 + \Gamma_2 \leq 1$.



This gives rise to

$$\delta^* \leq \frac{1 - \sum_{i=1}^M \Gamma_i^2}{\sum_{i=1}^M \Gamma_i T_j}.$$

For $M > 2$, with $T_1 \geq 1$, $T_2 \geq 2$, and $T_i \geq 3$ for $i \geq 3$, the right-hand side is bounded above by

$$g(\Gamma_1, \Gamma_2) \equiv \frac{1 - \Gamma_1^2 - \Gamma_2^2}{\Gamma_1 + 2\Gamma_2 + 3(1 - \Gamma_1 - \Gamma_2)},$$

subject to $\Gamma_1 + \Gamma_2 \leq 1$.

A plot of the function $g(\cdot, \cdot)$ is as shown in Figure 4.

The maximum is attained at $\Gamma_2 = 0.4$, $\Gamma_1 = 0.2$, with a value of $g(0.2, 0.4) = 0.4$. Hence, the generalized Homer's policy has a worst-case performance of 1.4.

THEOREM 5. For the multivendor delivery scheduling problem, when the intervals T_i are relatively prime, the peak inventory level attained by the generalized Homer's policy is at most 140% of the optimal peak inventory level.

5. Delivery Scheduling Problem

While a good policy for the inventory staggering problem can be used to construct a good policy for the delivery scheduling problem, the worst-case results obtained for the inventory staggering problem do not carry over to the delivery scheduling problem. In general, if \mathcal{P} is a $(1 + \theta)$ approximation algorithm for the inventory staggering problem, then we have

$$S^{\max}(\mathcal{P}) \leq (1 + \theta) S^{\max}(\mathcal{P}^*).$$

Hence,

$$\begin{aligned} S^{\max}(\mathcal{P}) - \frac{1}{2} \sum_{i,j} \gamma_{ij} T_i \\ \leq (1 + \theta) \left(S^{\max}(\mathcal{P}^*) - \frac{1}{2} \sum_{i,j} \gamma_{ij} T_i \right) + \theta \left(\frac{1}{2} \sum_{i,j} \gamma_{ij} T_i \right). \end{aligned}$$

So from Theorem 1,

$$\frac{1}{2} \sum_{i,j} \gamma_{ij} T_i - S^{\min}(Q) \leq (1 + \theta) \left(\frac{1}{2} \sum_{i,j} \gamma_{ij} T_i - S^{\min}(\mathcal{Q}^*) \right) + \theta \left(\frac{1}{2} \sum_{i,j} \gamma_{ij} T_i \right),$$

i.e., the policy \mathcal{Q} for the delivery scheduling problem is within $(1 + \theta)$ times of the optimal solution, plus an additive term of $\theta((1/2) \sum_{i,j} \gamma_{ij} T_i)$. From Theorem 1 and (3),

$$\frac{1}{2} \sum_{i,j} \gamma_{ij} T_i - S^{\min}(\mathcal{Q}^*) = S^{\max}(P) - \frac{1}{2} \sum_{i,j} \gamma_{ij} T_i \geq \frac{1}{2} \sum_{i,j} \gamma_{ij}^2 T_i.$$

Hence, we conclude that the policy \mathcal{Q} is a

$$1 + \theta \left(1 + \frac{\sum_{i,j} \gamma_{ij} T_i}{\sum_{i,j} \gamma_{ij}^2 T_i} \right)$$

approximation algorithm for the delivery scheduling problem. The bound is thus data dependent. It could be bad if the number of trucks used is large for each vendor. This is not surprising, because the lower bound for the delivery scheduling problem can be very bad in a situation when the deliveries from the vendors are clustered together in the optimal solution.

In general, finding a delivery scheduling policy with good (constant) worst-case bound is a challenging and exceedingly difficult problem. Nevertheless, the analysis performed in the previous section can be used to obtain insights on the behaviour of the (dual) generalized Homer’s policy for the delivery scheduling problem. This policy is natural and easy to implement in practice and does not involve coordination between vendors with different replenishment intervals. It involves only coordination of deliveries for vendors with the same replenishment intervals.

Consider the case in which there are only two distinct replenishment intervals. The worst-case error, based on our analysis (cf. (4)) for the two-vendor case, depends on the term

$$\theta = \max_{\alpha: 0 \leq \alpha \leq 1} \frac{2}{q/\alpha + p/(1-\alpha)},$$

and the relative magnitude of the terms $(1/2) \sum_{i,j} \gamma_{ij} T_i$ and $(1/2) \sum_{i,j} \gamma_{ij}^2 T_i$. For replenishment intervals T_1 and T_2 with large relatively prime factors of p and q , the term θ can be extremely small, and the generalized Homer’s policy is expected to be pretty good.

As an illustration of the usefulness of the generalized Homer’s policy, we consider the numerical example discussed earlier (as shown in Figure 1). Note that the delivery intervals for vendors A and B are identical. Assuming that each truck carries the same load, the generalized Homer’s policy will thus focus on coordinating the deliveries from these vendors and space out the trucks from both vendors

equally over the time horizon. Consider the following delivery schedule:

- Vendor A delivers in the morning, at 0900, 1020, 1140, and 1300, after which her schedule will repeat at eight-hour intervals.
- Vendor B delivers in the afternoon at 1420 and 1540, after which her schedule will repeat at eight-hour intervals.
- Vendor C’s delivery schedule remains the same as before, i.e., vendor C delivers at 0930, after which deliveries occur at four-hour intervals.

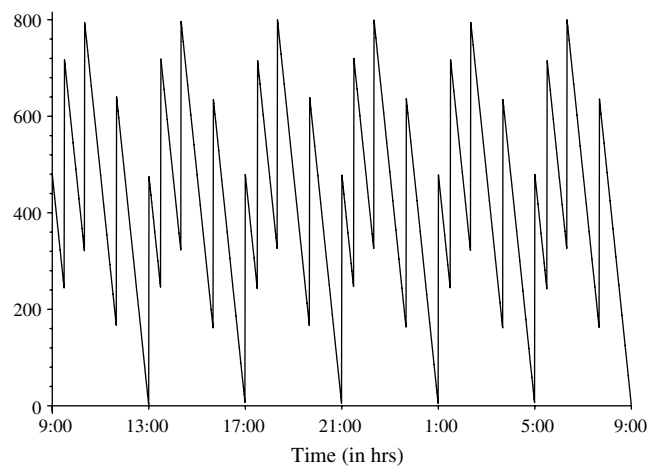
The new inventory level fluctuation is as shown in Figure 5. Note that unlike the previous case, the four trucks from vendor A are not spaced out at equal intervals of two hours each. Instead, the delivery interval between successive trucks are 1 hour 20 minutes (three times) and four hours (once). Nevertheless, each truck takes eight hours to complete a circuit.

With the delivery quantities remaining the same as in the last example, the average inventory level for this new schedule is only 420 units, a drop of 22% compared to the original schedule. Furthermore, the peak inventory level drops to 800 units (a reduction of 16.7%).

To further improve the performance of the delivery schedule, we need to next look at synchronizing the activities across vendors with different replenishment intervals. To this end, we note that the deliveries for vendor C (at time 0930, 1330, etc.) arrive at the factory when there are still excess inventory. To further reduce the inventory level we can push the deliveries from vendor C to a later time. To prevent stock-out at the factory, and keeping the delivery schedules from vendors A and B unchanged, the latest permissible delivery times for vendor C are 1000, 1400, etc. By this adjustment to the schedule, it is easy to see that the average inventory level drops by a further 6 units to 414 units.

By fine-tuning the schedules obtained from the generalized Homer’s policy, improvement in the delivery performance is possible, especially if the delivery intervals are

Figure 5. Inventory level fluctuation for the new schedule.



nested. Another possible way to improve the performance of the delivery scheduling problem is to lengthen the delivery intervals for some of the vendors. In this way, we need to choose the best combination of replenishment intervals for the vendors (subject to some lower bound constraints) so that the average inventory in the factory is minimum. Let $\{T'_i\}$ denote the optimal combination of the replenishment intervals, subject to $T'_i \geq T_i$, where T_i is the lower bound to the replenishment intervals. It is easy to see that in the optimal solution, $T'_i < 2T_i$. Otherwise, by replacing T'_i by $T'_i/2$, the average inventory level in the new schedule will be smaller. This observation allows us to restrict the possible values for T'_i in the range $[T_i, 2T_i - 1]$. The optimal combination of the replenishment parameters can thus be searched via enumerating over the possible values of T'_i .

6. Concluding Remarks

In this paper, we study the delivery scheduling problem faced by a factory in a multivendor JIT environment. We show that the problem is rich in structure. We establish a fundamental relationship between the peak, the minimum and the average inventory level in this class of problems, and use it to show that the delivery scheduling problem is equivalent to the classical inventory staggering problem. This connection appears to be interesting but nontrivial because this relates the time average objective function in the delivery scheduling problem to the min-max objective function in the inventory staggering problem.

We thus focus our attention on the inventory staggering problem. By examining the periodicity structure of the schedule using the Chinese remainder theorem, we show that the generalized Homer's policy is a good heuristic for this class of problems. In the case of two vendors, the generalized Homer's policy is at most 134.31% of the optimal. This result applies for all replenishment intervals, even if they are not relatively prime. For the general case, as long as the replenishment intervals are relatively prime, the generalized Homer's policy still performs relatively well, with a worst-case result of at most 140% of the optimal. The attained performance is somewhat surprising, because the generalized Homer's policy makes no attempt to synchronize the deliveries from vendors with different replenishment intervals. When the intervals are nested, proper coordination between clusters of vendors is crucial in improving worst-case results. This is indeed the approach taken by Teo et al. (1998), who obtained a complicated $15/8$ approximation algorithm for this case.

The problem considered in this paper is realistic but very difficult. Furthermore, our analysis shows (via a transformation to the inventory staggering problem) that the choice of the replenishment intervals, in stable and periodic delivery environment, can have profound impact on the average inventory level. A small change in the replenishment intervals can lead to significant change in the average inventory level attained. This is not clear a priori for the delivery

scheduling problem but is evident from the fact that the inventory staggering model is sensitive to the choice of the replenishment intervals. We note also that finding a good tractable approximation to the peak storage usage, for given replenishment intervals, is a common issue in the study of inventory system with storage consideration (e.g., SRMIS). Through the analysis presented in this paper, we hope to excite the larger research community in obtaining a more robust solution to this problem.

Note that our results obtained so far assume that the broad contract parameters have already been negotiated and the focus is on fine-tuning the delivery schedules for optimal inventory performance. A challenging problem is to optimize the choice of the load of delivery Γ_i , the number of trucks n_i used, and/or also the replenishment intervals T_i , so that average inventory in the factory can be further reduced. As another example, suppose the number of trucks n_i available is fixed. It is already difficult to determine the load Γ_i to be allocated to each vendor. It is not clear that allocating the entire load to the vendor with the shortest replenishment interval (i.e., single sourcing) will be the best solution for this case, because the inclusion of other vendors might allow the system to deploy more trucks that will help bring down the inventory level in the system. We leave the investigations of these problems for future research.

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