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Contests with endogenous entry

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Abstract This paper studies the effort-maximizing design of a complete-information contest with endogenous entry. A fixed pool of homogenous potential players with identical marginal bidding cost must incur an entry cost to enter the contest before they bid for prize(s). The designer can flexibly adjust the impact function of a generalized nested lottery contest and use a fixed budget to fund single or multiple prizes. Applying Dasgupta and Maskin (Rev Econ Stud 53(1):1–26, 1986), we establish the existence of symmetric equilibrium for all contest mechanisms concerned. A uniform upper bound for expected overall bids is identified for any eligible contest, assuming that potential bidders play symmetric equilibria. We show that the upper bound can be achieved through a Tullock contest with a single contingent prize, which adopts compatible bundles of success function and entry fees/subsidies. In particular, we identify the conditions under which the optimum can be achieved by solely setting the right discriminatory power in a Tullock contest with a single fixed prize. Finally, our analysis characterizes the optimal shortlisting rule, which reveals that the contest designer generally should exclude potential bidders to elicit higher bids.

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1 Introduction

Economic agents are often involved in contests; they expend costly and nonrefundable effort to compete for a limited number of prizes. A wide variety of economic activities exemplify such competitions, which include rent-seeking, lobbying, political campaigns, R&D races, competitive procurement, college admissions, ascent of organizational hierarchies, and movement within internal labor markets.

The vast wealth of literature on contests has delineated economic agents' strategic behaviors in contests from diverse perspectives and identified the various institutional elements in contest design that affect bidding incentives. Most existing studies, however, focus on a setting in which a fixed number (n) of bidders participate. These studies, under the "fixed-n paradigm," typically abstract away from the ex ante participation decisions of bidders and focus on their post-entry activities, assuming that the actual number of active participants is commonly known. Our study complements the existing literature by examining a setting in which a fixed pool of potential bidders strategically decide whether to participate in a contest and then sink their bids upon entering the contest.

In our setting, participation incurs a nontrivial (fixed) cost: It allows a bidder to participate and is unrelated to his chances of winning. Each bidder weighs his expected payoff in future competitions against the entry cost and participates if and only if the former (at least) offsets the latter. As noted by Konrad (2009), a bidder often bears a nontrivial (fixed) entry cost. For instance, to participate in an R&D race, a research company may need to acquire necessary laboratory equipment, gather project-specific information, or turn down other profitable tasks—while its chances of winning depend on subsequent creative input. A nontrivial entry cost may lead potential bidders to participate randomly. The actual number of participants thus remains uncertain. In our setting, participants do not observe the actual entry decision of their opponents and must take this uncertainty into account when placing their bids.[1]

This entry-bidding game exhibits distinctive characteristics, which significantly complicate the equilibrium analysis. First, it exemplifies a discontinuous game with two-dimensional action space (Dasgupta and Maskin 1986). With strategic and stochastic entry, each potential bidder's strategy involves two elements: (1) whether to enter, and (2) how to bid upon entering.[2] The game thus distinguishes itself from

[1] We assume that the entrants do not observe the number of entrants. We consider this to be a reasonable assumption in many scenarios. For example, in many open innovation challenges and crowdsourcing contests, it is difficult for a participant to observe other potential candidates' actual entry decisions. As we discuss in the conclusion, there is no loss of generality to restrict our attention to optimal design of contests that do not disclose the actual number of participants.

[2] The literature on contests (Baye et al. 1994; Alcalde and Dahm 2010) recognizes that a well-defined contest success function (e.g., a Tullock contest) can be discontinuous at its origin, i.e., when all bidders bid zero.

standard contests that are typically identified as unidimensional discontinuous games. Conventional approaches to establishing the existence of equilibria in contest models (e.g., Baye et al. 1994; Alcalde and Dahm 2010) or in auction games do not apply to our settings. [3] This novel setting entails the application of Dasgupta and Maskin's (1986) general theorem on multidimensional discontinuous games. We establish that a *symmetric entry-bidding equilibrium* exists under a wide class of contest technologies, while allowing for flexible prize structures that can be contingent on the actual number of entrants: Each potential bidder enters with the same probability and adopts the same (possibly mixed) bidding strategy upon entry.[4] To our knowledge, our analysis provides the first application of the existence theorem for multidimensional discontinuous games in the contest literature.

Second, endogenous and (possibly) stochastic entry leads to substantially more extensive strategic interactions in bidding competitions. Our analysis sheds light on two classical questions in the contest literature. First, to what extent can the equilibrium bidding strategies (conditional on their entry) be solved for explicitly? Second, under what conditions would equilibria that do (or do not) involve pure-strategy bidding exist?[5] To place his bid, a participant must form a rational belief about the competing bidders' entry patterns and take into account all possible contingencies that could be caused by the (endogenously determined) random entries. With this flavor, the general property of a bidder's overall expected payoff function cannot readily be discerned, and results from existing studies do not carry over. Within a framework of Tullock contest technology, our analysis reveals the nature of the strategic decision problem and identifies the sufficient conditions under which participants do (or do not) randomize their bids upon entry. In addition, we find that in our setting pure-strategy bidding can take place under a wider range of conditions, than in a standard contest with free entry.

Endogenous (and potentially stochastic) entry and its impact on post-entry bidding behavior pose new issues for contest design. Based on our equilibrium results, we explore optimal contest design in a framework of generalized nested lottery contests with a contingent prize allocation scheme. We investigate how relevant institutional elements of a nested Tullock contest, e.g., the discriminatory power of the success function and prize allocation scheme, would affect the bidding efficiency of the contest and achieve the optimum. Our analysis demonstrates the extensive strategic trade-offs triggered uniquely by endogenous and stochastic entry. The main findings are as follows:

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[3] We elaborate on this issue in Sect. 4.

[4] It should be noted that the two-dimensional (entry and bidding) game cannot be reduced to a single-dimensional setting in which a bidder's cost function includes a fixed component. In other words, the fixed entry cost in our setting must be distinguished from the bidding cost. In our setting, to the extent that no bidder enters the contest, the contest designer retains the prize and no bidder wins. To the extent that all bidders enter but submit zero bids, they stand an equal chance of winning the prize. However, in a typical single-dimensional (one-shot) setting, if all contestants stay inactive, i.e., submit zero bids, they still stand an equal chance of winning, although no (fixed) cost is incurred.

[5] It is well known that a bidder's payoff maximization problem becomes irregular when the contest success function is excessively elastic to effort, e.g., when the discriminatory parameter r in a Tullock contest exceeds certain boundaries. Endogenous and stochastic entry compounds the complexity.

- Focusing on the symmetric entry-bidding equilibria, we first identify a uniform "upper bound" for expected overall bid in all eligible contests considered in this paper.[6] This upper bound applies universally regardless of the prevailing winner selection mechanism (i.e., the contest success function) and prize structures.
- To achieve the upper bound, the contest mechanism induces potential bidders to enter the contest with a unique optimal probability, which depends on various key environmental factors such as the number of potential bidders and the size of entry cost. We show that optimal contests and their associated prize allocation schemes are not unique, and that the upper bound can be achieved under a continuum of contest mechanisms.[7] In particular, we demonstrate that the upper bound can always be implemented through Tullock contests with a single contingent prize when the designer is able to fine-tune the success function and supplement it with properly set entry fees/subsidies.[8] Further, we identify the conditions under which the optimum can be achieved by adjusting only the contest success function in a Tullock contest with a single fixed prize, i.e., setting the right discriminatory power.
- Focusing on a Tullock contest success function, we show that the optimally chosen discriminatory power r does not vary monotonically with key environmental factors, such as the number of potential bidders and the size of entry cost. The designer may paradoxically prefer a "noisier" winner selection mechanism, which provides a lower-power incentive in contests. The optimal contest may induce stochastic entry, which causes a greater r to give rise to competing effects at different layers. First, it gives rise to a trade-off between ex post bidding incentives and ex ante entry incentives. A greater r intensifies competition but also leaves lesser rent for participants and restricts bidders' entry. Second, there is tension between the level of participation and the incentive for individual bids. More active entry (under a smaller r) allows the contest to engage more bidders and tends to amplify their overall contributions. However, it also leads individual participants to bid more prudently, as they anticipate a lower chance of winning. To elicit the maximum overall bid, the contest may require a nosier technology that optimally balance these countervailing forces, in contrast to conventional wisdom in contests with exogenous and deterministic participation.
- The literature conventionally holds that the overall bid of a contest always increases with the number of bidders when their participation is deterministic. However, we demonstrate in our setting that contest designers may prefer to limit competition by inviting only a subset of potential bidders for participation; a contest may elicit lower bids when a larger pool of potential bidders is involved. We explore the functional relationship between the number of shortlisted potential bidders and the expected overall bid of the contest under properly set mechanisms and find

6 Not surprisingly, this upper bound of effort decreases with entry cost.

7 Contest designer can flexibly adjust the combination of a contest success function and matching prize allocation scheme. Either a single prize or multiple prizes may emerge as the optimum, depending on the contest success functions adopted by the designer.

8 The value of the single prize depends on the designer's initial budget and entry fees collected from entrants or subsidies paid to entrants.

it to be single-peaked. With this observation, we fully characterize the optimal shortlisting rule.[9]

The rest of the paper proceeds as follows. In Sect. 2, we discuss our paper's relation to the literature on contests. In Sect. 3, we set up a generic entry-bidding game and derive the upper bound of overall bid that can possibly be elicited in the game. Section 4 discusses the possibility of achieving the upper bound by structuring Tullock contests. In Sect. 5, we discuss several possible extensions to widen the scope of our analysis. Section 6 concludes the paper. All proofs are relegated to the Appendix.

2 Relation to the literature

Our paper complements the literature on contests in various ways.[10] It provides a formal account of equilibrium existence in the entry-bidding game for a large family of contest technologies, which includes Tullock contests as a subfamily and all-pay auctions as its limiting case. Szidarovszky and Okuguchi (1997) establish the existence of pure-strategy equilibria when contestants have concave production functions. The existence and properties of the equilibria remain a nagging problem for contests with less well-behaved technologies. Baye et al. (1994) establish the existence of mixed-strategy equilibria in two-player Tullock contests with $r \geq 2$. Alcalde and Dahm (2010) further the literature by showing that *all-pay auction equilibria* exist under a wide class of contest success functions.[11] Both studies apply the results of Dasgupta and Maskin (1986) on unidimensional discontinuous games. Our paper contributes to this literature by introducing bidders' entry decisions, while allowing the number of active bidders to be stochastic. These new flavors enrich our analysis by forming a two-dimensional discontinuous game and provide a novel application of the general result of Dasgupta and Maskin (1986) on multidimensional discontinuous games in the contest literature.

There is a growing literature on contests with endogenous entry. In their pioneering work, Higgins et al. (1985) study a tournament model in which each rent seeker bears a fixed entry cost and randomly participates in equilibrium. In an all-pay auction model with privately known entry costs, Kaplan and Sela (2010) provide a rationale for entry fees in contests. They let participants know who else has entered, while we focus mainly on uninformed participants.[12] Two recent experimental studies by Cason et al.

[9] We find that the maximum total effort from the optimally shortlisted players is also implementable in an asymmetric entry equilibrium of the original set of players.

[10] Our paper can also be related to the literature on standard oligopolistic competition; it echoes the arguments of Dixit and Shapiro (1986) and Shapiro (1989) on firms' behavior in oligopolistic markets. Shapiro (1989) shows that a Bertrand competition, which is fiercer, can be more anti-competitive *ex post* than a Cournot competition, which is *ex ante* more subdued, as the latter limits the contestability of the market and discourages entries. We focus on the issue of mechanism design in our particular context. In addition, the level of post-entry competition is a continuous variable and is considered to be a strategic choice of the contest designer.

[11] Wang (2010) also characterizes the equilibria in two-player asymmetric Tullock contests when r is large.

[12] We also discuss in the conclusion the ramifications of disclosure policy as an institutional element of contests.

(2010) and Morgan et al. (2012) also contribute to this research agenda by studying bidders' entries. Similar to Morgan et al. (2012) theoretical model, Fu and Lu (2010) also assume that potential bidders enter sequentially, so that neither setting involves stochastic participation. A handful of papers have examined contests with stochastic participation. The majority of these studies, however, assume exogenous entry patterns. Myerson and Wärneryd (2006) examine a contest with an infinite number of potential entrants whose entry follows a Poisson process. Münster (2006), Lim and Matros (2009) and Fu et al. (2011) assume a finite pool of potential contestants, with each contestant entering the contest with a fixed and independent probability.

Our study contributes to this literature by exploring the optimal contest design in a context with endogenous and potentially stochastic entries. First, our analysis complements the literature on the proper level of precision in evaluating bidding performance. Conventional wisdom says that a more precise evaluation mechanism incentivizes bidders. Gershkov et al. (2009) and Giebe and Schweinzer (2011) provide two novel applications of this principle, and both espouse the merits of a more precise contest as incentive devices. A handful of studies, however, espouse low-powered incentives in contests and demonstrate that a less "discriminatory" contest can improve efficiency. One salient example is provided by Lazear (1989), who argues that excessive competition leads to sabotage. A more popular stream in the literature instead stresses the handicapping effect of an imprecise performance evaluation mechanism in (two-player) asymmetric contests. When contestants differ in their abilities, a noisier contest balances the playing field. This effect encourages weaker contestants to bid more intensely and deters stronger ones from shirking. O'Keeffe et al. (1984) were among the first to formalize this logic. This rationale is further elaborated on by Che and Gale (1997, 2000), Fang (2002), Nti (2004), Amegashie (2009), and Wang (2010). In a recent study, Epstein et al. (2011) contend that contest designers still prefer all-pay auctions to Tullock contests if they can strategically discriminate between bidders. In contrast to these studies, our paper adopts a symmetric contest setting and stresses the trade-off between *ex post* bidding incentives and *ex ante* entry incentives. In this regard, it is more closely related to Cason, Masters, and Sheremeta's (2010) experimental study, which compares endogenous entries in all-pay auctions and lottery contests.

Our finding on optimal exclusion echoes a handful of pioneering studies by Baye et al. (1993), Taylor (1995), Fullerton and McAfee (1999), and Che and Gale (2003). These studies focus on heterogeneous contestants and concern themselves with selecting (usually two) players of the "right types." Our result, however, obtains in a setting of homogeneous players and concerns itself with creating a contest of the "right size." Dasgupta (1990) studies a two-stage procurement tournament, in which bidders invest in cost reduction in the first stage and place their bids in the second. Wider competition may diminish bidders' incentive to engage in R&D. Limiting the number of competing firms may or may not benefit the principal. None of these studies involves entry cost or endogenous entry. In contrast, an invited (potential) bidder in our setting decides whether to enter the subsequent contest, and the entry pattern in the equilibrium remains endogenous and potentially stochastic.

Bidders' endogenous entry has also been widely recognized as an important strategic element in auction games. Myerson (1981) studies a revenue-maximizing auction when bidders bear zero entry costs. Samuelson (1985), McAffee and McMillan (1987), Engelbrecht-Wiggans (1987, 1993), and Levin and Smith (1994) further study efficient and revenue-maximizing auctions when bidders have to bear some fixed entry costs. These studies typically conclude that the revenue-maximizing auction requires a lower reservation price than that of Myerson (1981) under endogenous entry: A lower reservation price weakens *ex post* incentive to bid, while it encourages the *ex ante* incentives for entry. The auction design problem addresses an adverse-selection problem, while our contest design problem primarily addresses a moral hazard problem. Not surprisingly, our finding on optimal adoption of discriminatory power in a Tullock contest only partially echoes insights from the auction literature: In our setting, the optimal design can involve an *ex ante* incentive that is either weaker (i.e., a smaller r) or stronger (i.e., a bigger r) than that in the free–entry benchmark.

Shortlisting and exclusion have long been an important theme in auctions with costly entry. In a setting with participation cost, Samuelson (1985) and Lu (2009) find that the optimal revenue can either increase or decrease with the number of potential bidders. In a setting with value discovery cost, Levin and Smith (1994) find that the optimal revenue must decrease with the number of potential bidders to the extent that optimal symmetric entry is in mixed strategy. Our setting resembles that of Sameulson (1985) and Lu (2009) and differs from that of Levin and Smith (1994) in the ways that the entry cost is modeled. However, our finding on the optimality of shortlisting echoes Levin and Smith (1994) while diverging from Lu (2009): The maximum total effort must drop with the number of potential players when optimal symmetric entry gets to be stochastic.

3 Contest with endogenous entry

This section proceeds in three steps. First, we set up a generic entry-bidding game, which allows for a wide class of contest technologies and a contingent prize allocation scheme. In this game, a fixed pool of potential bidders make costly and endogenous entry into a contest and compete for either a single or multiple (potentially contingent) prize. Second, we establish that a symmetric entry-bidding equilibrium always exists in this generic game. Third, focusing on such an equilibrium, we demonstrate that the overall bid in the contest, regardless of the prevailing winner selection mechanism, is subject to an upper limit.

3.1 Setup

A fixed pool of $M(\geq 2)$ identical risk-neutral potential bidders simultaneously decide whether to enter a contest. Each participant has to bear a fixed entry cost $\Delta > 0$. To accommodate bidders' endogenous entry, we allow for a very flexible prize structure, which can be contingent on the actual number of entrants N. Specially, when $N \in \{1, 2, ..., M\}$ players enter, they compete for a series of prizes $\{v_N^1, ..., v_N^n, ..., v_N^N\}$ with $v_N^1 \geq v_N^2 \geq ... \geq v_N^N \geq 0$. Each prize is allowed to be of zero value, but the total prize money is capped by a fixed sum $V(> 0)$, i.e., $\sum_{n=1}^{N} v_N^n \leq V$ for all possible N.

Each entrant is eligible for exactly one prize. When nobody enters, the prize money is kept by the contest designer.

The prizes are allocated to the entrants through a generalized multi-prize nested lottery contest (see Clark and Riis 1996 and 1998, and Fu and Lu 2012) with a ratio-form success function. Upon entry, participants simultaneously submit their bids x_i without observing the number of entrants. However, we assume that the number of entrants is verifiable *ex post*, and thus the prize scheme can be contracted on it. If there is only one entrant, he automatically wins the single prize v_1^1. Suppose $N \geq 2$ players indexed by $\{i_1, i_2, ..., i_N\}$ enter and their bidding entries are $\mathbf{x}_N = (x_{i_1}, x_{i_2}, ..., x_{i_N})$. Let $(j_1, j_2, ..., j_N)$ denote an arbitrary prize allocation outcome, i.e., player $j_n \in \{i_1, i_2, ..., i_N\}$ wins the n-th highest prize v_N^n, $n = 1, 2, ..., N$. If all entrants place strictly positive bids, an allocation outcome $(j_1, j_2, ..., j_N)$ occurs with a probability

$$p_N(j_1, j_2, ..., j_N) = \sum_{n=1}^{N} \frac{g(x_{j_n})}{\sum_{n'=n}^{N} g(x_{j_{n'}})}, \tag{1}$$

where $g(\cdot)$ is conventionally called the impact function of the contest mechanism. The function $g(\cdot)$ is assumed to be continuous with $g(0) = 0$ and $g'(\cdot) > 0$. The prize-allocation mechanism builds on the widely used ratio-form contest success function, which is axiomatized by Skaperdas (1996) and Clark and Riis (1998). Fu and Lu (2012) further provide a micro-foundation for this multi-prize contest success function (1) from a noisy ranking perspective.[13] This family of contest technologies includes nested Tullock contests as an important and salient subclass, with $g(x_i) = x_i^r$, $r > 0$. When $r \to \infty$, all-pay auctions are obtained as a limiting case.

Suppose that $N_0 \leq N$ entrants place zero bids. They must win each of the lowest N_0 prizes with equal probability $\frac{1}{N_0}$, while those with positive bid win the first $N - N_0$ highest prizes. Let $(j_1, j_2, ..., j_{N-N_0})$ denote an arbitrary prize allocation outcome among the players with positive bids, i.e., player $j_n \in \{j_1, j_2, ..., j_{N-N_0}\} \subset \{i_1, i_2, ..., i_N\}$ wins the nth highest prize v_N^n, $n = 1, 2, ..., N - N_0$. The probability that $(j_1, j_2, ..., j_{N-N_0})$ occurs is

$$p_{N-N_0}(j_1, j_2, ..., j_{N-N_0}) = \sum_{n=1}^{N-N_0} \frac{g(x_{j_n})}{\sum_{n'=n}^{N-N_0} g(x_{j_{n'}})}.$$

A potential bidder i's strategy is denoted by an ordered pair $(q_i, \mu_i(x_i))$, where q_i is the probability of his entering the contest and the probability distribution $\mu_i(x_i)$ depicts his bidding strategy conditional on his entry. The distribution $\mu_i(x_i)$ reduces to a singleton when the participant does not randomize his bids.

When bidding x_i, a participating bidder i bears a cost $c(x_i)$. A participant i's expected payoff is simply the expected amount of prize money he can win net of his bidding cost.

13 Using a similar approach, Clark and Riis (1998) reveal the microeconomic underpinning of its single-prize case.

We assume that participants do not observe others' actual moves in the entry stage. Each participant bids based on his belief about others' entry and bidding strategies, which is derived by Bayes' rule. Each player's entry decision depends on his expected payoff upon entry. We adopt the notion of Bayesian Nash equilibrium to solve the game. An equilibrium is a strategy combination $\times_{i=1}^{M}(q_i, \mu_i(x_i))$ of all contestants, which requires that the pair strategy $(q_i, \mu_i(x_i))$ of each potential bidder i maximize his expected payoff based on his belief based on Bayes' rule and others' strategy profiles $\times_{j\neq i}(q_j, \mu_j(x_j))$. In this paper, we restrict our attention to bidding strategies $\mu_i(x_i)$ that are independent of the entry probability q_i.

Because of the symmetry among bidders, throughout the entire analysis we focus on symmetric equilibria in which all potential bidders play the same strategy $(q^*, \mu^*(x))$. The existence of a symmetric equilibrium for a general impact function $g(\cdot)$ will be established in Sect. 3.3. The issue of asymmetric equilibrium will be discussed in Sects. 3.3 and 5.2.

We begin with a family of contest technologies as described by (1). It should be noted, however, that our main analyses—e.g., the result of equilibrium existence in Sect. 3.3, those in Sect. 3.4 on the maximal bid of generic entry-bidding game, and those in Sect. 4 on the optimal contest design—can readily be extended to settings with other contest technologies. The main implications do not rely on the specific form of ratio-form contest success function.

3.2 Some preliminaries

The following assumption focuses our analysis on the most relevant case, since no entry is triggered if it costs more than the winner's purse.

Assumption 1 $\Delta < V$.

We define a critical cutoff probability, which is used repeatedly throughout the analysis.

Definition 1 Let $q_0 \in (0, 1)$ be the unique solution to $(1-q)^{M-1}V - \Delta = 0$.

The cutoff q_0 defines a lower bound for entry probability in a contest with a single prize V for any contest technology. Such a contest must induce players to enter with a probability higher than q_0 in a symmetric equilibrium. To see the logic, suppose that in the equilibrium, every player enters with probability $q \in (0, q_0)$. An entrant can secure a payoff $(1-q)^{M-1}V - \Delta > 0$ from bidding zero when all other potential bidders stay out of the contest, which occurs with a probability $(1-q)^{M-1}$. This conflicts, however, with the fact that his payoff should be zero in a symmetric equilibrium with random entry.

3.3 Existence of symmetric equilibrium

The entry-bidding game has two main features. First, the strategy of each player involves two elements: entry and bidding. Second, a potential bidder's payoff can be discontinuous, since the contest success function can be discontinuous. For instance, the payoff function is discontinuous at the origin (see Baye et al. 1994, and Alcalde and Dahm 2010), i.e., when all participants bid zero.

The conventional approach (in auctions with endogenous entry) to establishing the existence of symmetric equilibria proceeds in two steps, which disentangles the two elements in each player's strategy and simplifies the analysis. In the first step, potential bidders are assumed to enter the competition with given (symmetric) entry probabilities. One establishes the existence of symmetric bidding equilibrium under each given entry probability q. Bidders' equilibrium payoff function in the auction $\pi(q)$ is obtained accordingly, which is typically continuous and monotonic in q. In the second step, a (typically unique) entry probability is identified, which equalizes a potential bidder's expected payoff in the auction $\pi(q)$ and his entry cost.

As will be further illustrated in Sect. 4.1, this disentangling approach has only limited utility in our setting. The reasons are twofold. First, the existence of bidding equilibrium in a contest conventionally relies on Dasgupta and Maskin's (1986) theorem on unidimensional games. This theorem, however, does not apply to games with an uncertain number of players. Thus, the general existence of a symmetric bidding equilibrium under a given entry probability q cannot readily be verified. Second, a contest game with a fixed entry probability may not be well behaved, and the bidding strategy is not universally solvable (as will be seen in Sect. 4.1). It remains difficult to characterize the properties (e.g., continuity and monotonicity) of bidders' expected payoffs, even if an equilibrium exists.

As a result, the entry-bidding game in general should be recognized as a two-dimensional discontinuous game, and thus it entails the application of Dasgupta and Maskin's (1986) equilibrium-existence theorem for discontinuous games with multi-dimensional strategy space.

Theorem 1 *(a) For any impact function* $g(\cdot)$ *with* $g(0) = 0$ *and* $g'(\cdot) > 0$*, a symmetric equilibrium* $(q^*, \mu^*(x))$ *always exists. In the equilibrium, each potential bidder enters with a probability* $q^* \in (0, 1]$ *and his bid follows a probability distribution* $\mu^*(x)$.

(b) If $q^* \in (0, 1)$*, each potential bidder receives an expected payoff of zero in the entry-bidding equilibrium. If* $q^* = 1$*, a representative bidder's equilibrium payoff must be nonnegative.*

To our knowledge, Theorem 1 and its proof provide the first application of Dasgupta and Maskin's (1986) equilibrium-existence result on two-dimensional discontinuous games in the literature on contests. Two remarks are in order.

First, the equilibrium-existence result applies to broader contexts. We explicitly adopt ratio-form contest success functions to economize on our presentation and facilitate subsequent discussion on contest design. However, the proof does not rely on the specific properties of this contest mechanism. The analysis can readily be adapted to contests with more broadly defined prize allocation mechanisms, such as those in Alcalde and Dahm (2010), by redefining the discontinuity set slightly.

Second, it should be noted that asymmetric equilibria always exist in the entry-bidding game, in which a subset of potential bidders stays inactive regardless, while the others enter either randomly or deterministically. We mainly focus on the symmetric equilibrium primarily because a symmetric equilibrium, in which all players remain active, can arguably be viewed as a natural focal point of the game. Symmetric equilibrium has conventionally been the focus of the majority of studies, both theoretical and experimental, in contest or auction games with symmetric players (see Kuzmics et al. 2013). As Noussair and Silver (2006) and Barut et al. (2002) point out, "a symmetric equilibrium is the most plausible equilibrium that might emerge in a symmetric setting," as "no obvious means to coordinate on an asymmetric equilibrium," given that players have symmetric positions and their moves are simultaneous. Kuzmics et al. (2013) contend that players, who are in symmetric positions, "may have no cues or labels outside the description of the game, as given, that could serve, via a norm or convention, to coordinate their actions on an asymmetric outcome."[14]

3.4 Upper bound of overall bids

Central to contest literature is the question of how the institutional elements of the contest affect bidding efficiency. As Gradstein and Konrad (1999) argue, "Contest structures result from the careful consideration of a variety of objectives, one of which is to maximize the effort of contenders." We hereby demonstrate that the total bid from each given contest is subject to a uniform upper bound, which provides the maximum amount of overall bid a contest could possibly elicit, regardless of the governing winner selection mechanism and prevailing prize allocation rule. We further impose a regularity condition on the bidding cost function.

Assumption 2 The bidding cost function $c(x_i) = x_i^{\alpha}$ is convex, with $\alpha \geq 1$.

Convex bidding cost is commonplace in contest literature. The equilibrium existence theorem does not rely on the assumption. It allows, however, for a more regular and tractable analysis when characterizing equilibrium bidding behaviors and searching for optimal contest structures.

Suppose that potential bidders enter with a probability $q \in (0, 1]$ in a symmetric equilibrium. When N of them participate, they compete for a series of prizes $\{v_N^1, ..., v_N^n, ..., v_N^N\}$. As mentioned above, the sum of these prizes is capped by an upper limit $V(> 0)$, i.e., $\sum_{n=1}^{N} v_N^n \leq V$, where V is the contest designer's budget constraint for prize money. As a result, the rent to be dissipated in the contest cannot exceed $[1 - (1 - q)^M]V$.[15] Hence, bidders receive an expected overall rent less than $[1 - (1 - q)^M]V$, while they incur, on average, entry cost $Mq\Delta$. The following fundamental equality must hold in a symmetric equilibrium, as the expected total payoff of players must be positive:

$$[1 - (1 - q)^M]V \geq Mq(\Delta + E(x^{\alpha})), \tag{2}$$

[14] To our knowledge, there exist no systematic studies of the robustness of symmetric vis-à-vis asymmetric plays in experiments when players are symmetric and the game has multiple equilibria. Huck et al. (2002) show in a sequential-move duopoly setting that a symmetric strategy profile (Cournot outcome) is played by subjects, although the theory predicts a unique asymmetric equilibrium (Stakelberg outcome).

[15] A detailed proof is available from the authors.

where $E(x^{\alpha})$ denotes the equilibrium expected effort costs of an entrant. An upper bound for equilibrium expected overall bidding cost can then be identified as follows:

$$MqE(x^{\alpha}) \leq [1-(1-q)^{M}]V - Mq\Delta. \tag{3}$$

The convexity of cost function ($\alpha \geq 1$) further implies the following important inequality that bounds the expected overall bid $(MqE(x))$ from above:

$$(MqE(x)) = MqE[(x^{\alpha})^{\frac{1}{\alpha}}] \leq Mq[E(x^{\alpha})]^{\frac{1}{\alpha}}. \tag{4}$$

By (3) and (4), we further obtain

$$(MqE(x)) \leq [Mq]^{\frac{\alpha-1}{\alpha}}\{[1-(1-q)^{M}]V - Mq\Delta\}^{\frac{1}{\alpha}}. \tag{5}$$

Apparently, an equality holds if and only if the behavioral bidding strategy upon entry does not involve randomization, except for the knife-edge case of $\alpha = 1$. The inequality (5) yields important implications: Regardless of the equilibrium bidding strategy upon entry, the expected overall bids that a contest could elicit in a symmetric equilibrium with entry probability q will never exceed the upper limit as given by RHS of (5). Denote this upper bound by

$$\bar{x}_{T}(q) \triangleq (Mq)^{\frac{\alpha-1}{\alpha}}\left\{[1-(1-q)^{M}]V - Mq\Delta\right\}^{\frac{1}{\alpha}}, \tag{6}$$

with $q \in (0, 1]$. We further define the following cutoff.

Definition 2 Let $\bar{q} = \arg\max_{(1-(1-q)^{M})V \geq Mq\Delta, q\in[0,1]}\{q\}$.

Note that when $\frac{V}{M} < \Delta$, $\bar{q} < 1$; when $\frac{V}{M} \geq \Delta$, $\bar{q} = 1$. It is clear that the entry-bidding game cannot trigger an equilibrium, in which all potential bidders enter with a probability more than $\bar{q}$: They would otherwise end up with negative expected payoff even if they exert zero effort.

Lemma 1 $q_0 < \bar{q}$.

The function $\bar{x}_T(q)$ has the following important properties.

Theorem 2 *(a) The function $\bar{x}_T(q)$ is single-peaked at a cutoff $\hat{q} \in (q_0, 1]$. Specifically, if $\frac{V}{M} < \frac{\alpha}{\alpha-1}\Delta$, then $\hat{q} \in (q_0, \bar{q})$; if $\frac{V}{M} \geq \frac{\alpha}{\alpha-1}\Delta$, then $\hat{q} = 1$. Moreover, $\hat{q}$ decreases with M.*

(b) The function $\bar{x}_T(q)$ strictly increases with q when $q \in (0, \hat{q})$, and strictly decreases when $q \in (\hat{q}, 1]$.

The convex bidding cost function ensures that the function $\bar{x}_T(\cdot)$ is well behaved: As shown by Theorem 2, it is single-peaked and uniquely maximized by $\hat{q} \in (q_0, 1]$. By the theorem, in an entry-bidding game with M players, there is a uniform upper bound $\bar{x}_T(\hat{q})$ for the expected overall bid that can possibly be elicited from a symmetric equilibrium, regardless of the prevailing contest mechanisms.

Definition 3 Define $\bar{x}_T^* \equiv \bar{x}_T(\hat{q})$, which indicates the maximum amount of the overall bid a contest can elicit in a symmetric-entry equilibrium.

The upper bound $\bar{x}_T^*$ can be achieved in a contest if and only if the following four conditions hold.

1. The contest induces an equilibrium entry probability of exactly $\hat{q}$.
2. A total prize purse of V is distributed as long as some contestant(s) enter.
3. Each entrant ends up with a zero expected payoff.[16]
4. The contest must lead bidders to play a pure bidding strategy upon entry except for the knife-edge case of $\alpha = 1$.

Hence, the contest design problem ultimately boils down to one question: Can we find winner selection mechanism and prize structure to organize a contest that satisfies the above four conditions? The question is answered in the next section.

Before we proceed, we highlight the following corollary, which immediately arises out of Theorem 2(a).

Corollary 1 *When the bidding cost function is linear, i.e.,* $\alpha = 1$*, we have* $\hat{q} \in (q_0, \bar{q})$.

Corollary 1 states that with linear bidding cost, a contest achieves the upper bound $\bar{x}_T^*$ only if it induces random entry.

4 Achieving $\bar{x}_T^*$ by Tullock Contests

We now explore the optimal contest mechanism. We demonstrate that the upper bound $\bar{x}_T^*$ can always be achieved in a well-structured Tullock contest with a proper prize allocation scheme.

We consider three main instruments in designing the contest. A total budget $V > 0$ is at the contest designer's disposal. First, assuming that she concentrates her entire prize budget V to a single prize, we let her choose the discriminatory power parameter r. We show that a properly chosen discriminatory power parameter r alone can be adequate to achieve the objective in a wide range of settings. Second, we consider entry fees/subsidies as an auxiliary instrument. A Tullock contest can always achieve the upper bound $\bar{x}_T^*$ if the designer appropriately sets entry fees/subsidies, which are are collected from entrants and used to augment the winner's prize. Finally, we consider the ramifications of multiple prizes.

In a single-prize Tullock contest with $N \geq 2$ contestants, the probability of a participating bidder i's winning the prize is given by

$$p_N(x_i, \mathbf{x}_{-i}) = \frac{x_i^r}{\sum_{j=1}^N x_j^r} \text{ if } \sum_{j=1}^N x_j^r > 0, \quad \text{with } r > 0. \tag{7}$$

[16] This property automatically holds if property 1 holds and $\hat{q} < 1$.

4.1 Contest design by discriminatory power parameter r

We now explore the optimal contest mechanism that achieves the upper bound $\bar{x}_T^*$ solely through choosing an appropriate discriminatory power r of the contest. The designer chooses the size of r and announces it to potential bidders as part of the contest rules. An entry-bidding game ensues.

As shown by many existing studies, bidders' performance in a Tullock contest responds sensitively to the size of the power parameter r. The parameter reflects how effectively one's bid can be translated into win likelihood. The parameter thus has conventionally been viewed as a measure of the level of precision, or discriminatory power, of the winner selection mechanism in the contest literature (e.g., Nti (2004) and Gershkov et al. 2009). A greater r implies that the winner selection mechanism is related more closely to bids and less to various noisy factors. The level of precision or uncertainty in a contest can largely be manipulated as a strategic choice of the contest designer to achieve her objective. For instance, the designer can modify the judging criteria of the contest to suit her strategic goals, e.g., adjusting the weights of subjective components in contenders' overall ratings or varying the mix between experts and non-experts on judging committees in dancing contests (see Amegashie 2009). Alternatively, the International Table Tennis Federation (ITTF) in 2000 increases ball size: A larger ball discounts players' hitting power and reduces the velocities of both its movement and spin, thereby slowing down the match. It is widely held that the change substantially heightens the uncertainty in table tennis matches, as this weakens the role played by superior skills or strength in determining the winner. Conversely, the governing bodies of various sports have accepted Hawk-Eye, a complex computer system, as a technological means of adjudication since 2006. Hawk-Eye, "visually tracks the trajectory of the ball and displays a record of its statistically most likely path as a moving image". The system provides impartial opinions and largely reduces human errors in judgment. Alternatively, as pointed out by Gershkov et al. (2009), an organization's monitoring effort largely affects the quality of the information regarding workers' effort, thereby determining the precision of performance ranking. Moreover, in a procurement contest, the precision of quality ranking between competing products can largely be subject to the technical expertise of the procurement officer.

It is well known that a greater r increases the marginal return of a bid, thereby incentivizing bidders in a symmetric contest with deterministic entry. This conventional wisdom, however, is no longer straightforward in our context. As stated in the Introduction, endogenous and stochastic entry gives rise to competing forces at two levels. It remains *ex ante* unclear how the performance of the contest responds to varying r. In addition, these trade-offs would be further complicated in the presence of convex bidding costs. On the one hand, to elicit more bids, the optimal mechanism must avoid excessive rent dissipation caused by costly entry, which encourages the contest designer to limit entry. On the other hand, convex cost functions lead participants to refrain from aggressive bidding. The contest designer is thus compelled to engage more participants to even out their outlays, which mitigates the negative impact of increasing marginal cost on each active bidder. The optimum must balance these competing forces.

In what follows, we first explore how bidders' equilibrium behaviors respond to the structural elements of the contest. We identify the conditions under which pure-strategy bidding would (or would not) emerge, and solve for the equilibrium accordingly. The results allow us to identify the optimal r^*.[17]

4.1.1 Equilibrium analysis

The following lemma depicts the main properties of a symmetric equilibrium with pure-strategy bidding, if it exists.

Lemma 2 *Suppose that a symmetric equilibrium (q^*, x^*) with pure-strategy bidding exists. In such an equilibrium, the entry probability $q^* \in (0, 1]$ must satisfy*

$$\sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N} \frac{V}{N}\left(1 - \frac{N-1}{N}\frac{r}{\alpha}\right) \begin{cases} = \Delta, & \text{if } q^* < 1; \\ \geq \Delta & \text{if } q^* = 1. \end{cases} \quad (8)$$

Each participating bidder places a bid $x^* = \left[\sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N} \frac{N-1}{N^2} \frac{rV}{\alpha}\right]^{\frac{1}{\alpha}}$. The expected overall bid of the contest obtains as $x_T^* = Mq^*x^* = Mq^*\left[\sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N} \frac{N-1}{N^2}\frac{rV}{\alpha}\right]^{\frac{1}{\alpha}}$.

We call equation (8) the *break-even condition* of the entry-bidding game with pure-strategy bidding, which yields the following.

Lemma 3 *(a) For any $r > 0$, there exists a unique $q^* \in (0, 1]$ that satisfies the break-even condition (8). Hence, x^* is also uniquely determined for any given r.*
(b) The probability q^* strictly decreases with r when $q^* < 1$.

Lemma 3 establishes a unique correspondence between r and (q^*, x^*). The symmetric equilibrium with pure-strategy bidding must be unique for each given r whenever it exists. However, the strategy profile (q^*, x^*) in Lemma 2 may not constitute an equilibrium. Consider an arbitrary participating bidder i's payoff-maximization problem: When all other bidders play the strategy (q^*, x^*), he has an expected payoff

$$\tilde{\pi}_i(x_i) = \pi_i(x_i|q^*, x^*) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N} \pi_i^N(x_i), \quad (9)$$

[17] Although the upper bound $\bar{x}_T^*$ can be achieved in equilibria with mixed-strategy bidding when the bidding cost function is linear, we focus on its implementation through equilibria with pure-strategy bidding. The literature has provided little insight into bidding behaviors in mixed-strategy equilibria, e.g., in a Tullock contest with a large but finite r. Due to the lack of handy solutions, it is analytically difficult in the current context either to explicitly solve for an equilibrium when it involves mixed-strategy bidding or to verify the existence of an equilibrium with mixed-strategy bidding under a particular entry pattern. This analytical difficulty thus prevents us from identifying the correspondence between prevailing contest structures and the resultant equilibrium behavior.

with $\pi_i^N(x_i) = \frac{x_i^r}{x_i^r + (N-1)x^{*r}} V - x_i^\alpha$. The strategy profile (q^*, x^*) constitutes an equilibrium if and only if x^* is a global maximizer of $\tilde{\pi}_i(x_i)$. As is well known in the literature, when r exceeds one the function $\pi_i^N(x_i)$ and the weighted sum $\tilde{\pi}_i(x_i)$ are no longer be globally concave. The solution x^* thus cannot be established as the global maximizer without establishing discernible regularity of the function $\tilde{\pi}_i(x_i)$[18].

Define

$$\bar{r} = \left(1 + \frac{1}{M-2}\right)\alpha, \quad \forall M \geq 2.$$

The following sufficient condition is established.

Lemma 4 *When $r \in (1, \bar{r}]$, the function $\tilde{\pi}_i(x)$ is single-peaked for $x_i > 0$, and x^* is the unique interior local maximizer of $\tilde{\pi}_i(x_i)$ over $(0, \infty)$.*

To establish x^* as the global maximizer, a *boundary condition* $\tilde{\pi}_i(x^*) \geq \tilde{\pi}_i(0)$ has to be satisfied as well. By the break-even condition, a participating bidder's expected payoff in the contest when bidding x^*, i.e., $\tilde{\pi}_i(x^*)$, would amount to exactly Δ if $q^* < 1$ and $\tilde{\pi}_i(x^*) \geq \Delta$ if $q^* = 1$. However, an entrant can receive exactly a reserve payoff $(1-q^*)^{M-1}V$ from the contest by bidding zero: He wins V if and only if all other potential bidders stay out of the contest, which leads to a rent of $(1-q^*)^{M-1}V$. The bidder has an incentive to bid x^* if and only if $(1-q^*)^{M-1}V \leq \Delta$ when $q^* < 1$, and if and only if $\tilde{\pi}_i(x^*) \geq 0$ when $q^* = 1$.

Recall the cutoff q_0 depicted by Definition 1, which uniquely satisfies $(1-q_0)^{M-1}V = \Delta$. The discussion following Definition 1 reveals that in any symmetric equilibrium, the equilibrium entry probability q^* never falls below q_0. From the break-even condition (8), we define the following cutoff of r.[19]

Definition 4 Define $r_0 \in (\alpha(1+\frac{1}{M-1}), 2\alpha]$ to be the unique solution to $\sum_{N=1}^{M} C_{M-1}^{N-1} q_0^{N-1}(1-q_0)^{M-N}\frac{V}{N}(1-\frac{N-1}{N}\frac{r_0}{\alpha}) = \Delta$.

By Lemma 3(b), the equation $\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}\frac{V}{N}(1-\frac{N-1}{N}\frac{r}{\alpha}) = \Delta$ requires that the corresponding q fall below q_0 if $r > r_0$. Suppose that an $r > r_0$ induces pure-strategy bidding; the corresponding equilibrium entry probability must then be strictly less than q_0. However, by the discussion that follows Definition 1, the equilibrium entry probability in any symmetric equilibrium cannot be less than q_0. Based on this logic, we conclude the following.

Theorem 3 *A symmetric equilibrium with pure-strategy bidding does not exist if $r > r_0$. In the symmetric equilibria of the entry-bidding game, participating bidders must randomize their bids upon entry.*

Similar to contests with deterministic participation, pure-strategy bidding cannot be sustained when r is excessively large. Theorem 3 provides a sufficient condition under which randomized bidding must occur in this entry-bidding game.

[18] $\bar{r} = +\infty$, when $M = 2$.

[19] The proof for $r_0 \in \left(\alpha\left(1 + \frac{1}{M-1}\right), 2\alpha\right]$ is straightforward and thus omitted.

Note that for any $q \geq q_0$, an entrant would get a payoff $(1-q)^{M-1}V \leq \Delta$ by bidding zero. Suppose that r is bounded from above by both r_0 and $\bar{r}$. If $\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}\frac{V}{N}(1-\frac{N-1}{N}\frac{r}{\alpha}) \geq \Delta$ for $q = 1$, then $q^* = 1$ and the corresponding x^*, as defined by Lemma 2, must constitute the unique symmetric equilibrium by Lemma 3(b). If $\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}\frac{V}{N}(1-\frac{N-1}{N}\frac{r}{\alpha}) < \Delta$ for $q = 1$, then there must exist a unique $q^* \in [q_0, 1)$, which satisfies $\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}\frac{V}{N}(1-\frac{N-1}{N}\frac{r}{\alpha}) = \Delta$. Note that $q^* \geq q_0$ because $r \leq r_0$. In both cases, Lemma 4 implies that x^* must be the global maximizer to $\tilde{\pi}_i(x_i)$, which establishes the strategy profile (q^*, x^*) as the unique symmetric equilibrium with pure-strategy bidding. This is formally stated as follows.

Theorem 4 *For each $r \in (0, \min\{r_0, \bar{r}\}]$, the strategy profile (q^*, x^*), as characterized by Lemma 2, constitutes the unique symmetric equilibrium with pure-strategy bidding of the entry-bidding game.*

Theorem 4 imposes a (conservative) upper limit on r for pure-strategy bidding. Note that the condition $r \leq \bar{r}$ is sufficient but not necessary to establish x^* as the local maximizer of $\tilde{\pi}_i(x_i)$ for $x_i > 0$. It remains obscure how the equilibrium would behave if $\bar{r} < r_0$ and $r \in (\bar{r}, r_0]$. These concerns, however, can be dismissed when the number of potential bidders is small.

Corollary 2 *When M is small, i.e., $M \leq 3$, a symmetric equilibrium with pure-strategy bidding exists if and only if $r \leq r_0$.*

In these instances, $\bar{r} > r_0$ holds regardless of V and Δ, which guarantees that x^* maximizes $\tilde{\pi}_i(x_i)$ for $x > 0$. When $M \geq 4$, the cutoff r_0 may exceed $\bar{r}$. Analytical difficulty prevents us from fully characterizing the property of $\tilde{\pi}_i(x_i)$ when r exceeds $\bar{r}$. We leave this to our future research.

When $\frac{V}{M} < \Delta$, potential bidders must enter the contest randomly for any $r > 0$. In the case of $\frac{V}{M} \geq \Delta$, potential bidders may enter the contest either randomly or deterministically,[20] depending on the particular r. The next corollary focuses the situations of $\frac{V}{M} \geq \Delta$ and characterizes the condition for pure-strategy bidding with deterministic vs. random entry.

Corollary 3 *Assume $\frac{V}{M} \geq \Delta$. There exists a cutoff $\frac{\alpha M}{M-1}(1-\frac{M\Delta}{V}) \leq \min\{r_0, \bar{r}\}$. When $r \leq \frac{\alpha M}{M-1}(1-\frac{M\Delta}{V})$, there exists a unique symmetric equilibrium, in which all potential bidders enter the contest with probability one and place their bids in pure strategy upon entry. When $r > \frac{\alpha M}{M-1}(1-\frac{M\Delta}{V})$, there exists a unique symmetric equilibrium, in which all potential bidders enter the contest with probability $q^* < 1$ and place their bids in pure strategy upon entry.*

4.1.2 Optimal contest

Recall first Theorem 2, which states that the upper bound $\bar{x}_T^*$ is achieved when potential bidders enter with a probability $\hat{q}$ and place their bids in pure strategy. The theorem

20 In a contest with M participants, the pure strategy equilibrium payoff of each contestant is $\frac{V}{M}\left(1-\frac{M-1}{M}\frac{r}{\alpha}\right)$, he will enter with probability one if and only if $\frac{V}{M}\left(1-\frac{M-1}{M}\frac{r}{\alpha}\right) \geq \Delta$.

further shows that achieving this objective may require either random or stochastic entry, depending on the specific environment, i.e., $\hat{q} < 1$ if $\frac{V}{M} < \frac{\alpha}{\alpha-1}\Delta$, and $\hat{q} = 1$ if $\frac{V}{M} \geq \frac{\alpha}{\alpha-1}\Delta$.

In the latter scenario, the optimal contest is straightforward. By Corollary 3, for any $r \leq \frac{\alpha M}{M-1}(1 - \frac{M\Delta}{V})$, potential bidders enter the contest with probability one and place their bids in pure strategy. The following can be obtained.

Theorem 5 *Suppose* $\frac{V}{M} \geq \frac{\alpha}{\alpha-1}\Delta$. *The contest designer can elicit the upper bound* $\bar{x}_T^*$ *by setting* $r^* = \frac{\alpha M}{M-1}(1 - \frac{M\Delta}{V})$, *which induces a symmetric equilibrium with* $q^* = \hat{q} = 1$ *and pure-strategy bidding.*

This claim is straightforward. With $r \leq \frac{\alpha M}{M-1}(1 - \frac{M\Delta}{V})$, entry decision is trivial: Given $\frac{V}{M} \geq \frac{\alpha}{\alpha-1}\Delta$ and a small r, the contest leaves sufficient rent to attract all potential bidders to enter regardless. A greater r always provides a greater incentive to bid when players enter deterministically, and the contest ends up with a greater amount of bid when r rises. The contest designer, however, should not raise r above this level, as that would lead to suboptimal random entry with $q^* < \hat{q} = 1$. In this case, the contest can reach its optimum simply by adjusting r and setting it to the optimal level of $\frac{\alpha M}{M-1}(1 - \frac{M\Delta}{V})$.

We proceed to the scenario of $\frac{V}{M} < \frac{\alpha}{\alpha-1}\Delta$, which requires that potential bidders enter randomly in the optimum.

Definition 5 Let $r(\hat{q})$ be the unique solution of r to the break-even condition

$$V \sum_{N=1}^{M} C_{M-1}^{N-1} \hat{q}^{N-1} (1-\hat{q})^{M-N} \left[\frac{1}{N} - \frac{N-1}{N^2} \frac{r}{\alpha} \right] = \Delta, \quad \forall \hat{q} \in (q_0, \bar{q}). \tag{10}$$

Recall that when $\frac{V}{M} < \Delta$, $\bar{q} < 1$; when $\frac{V}{M} \geq \Delta$, $\bar{q} = 1$. Thus $r(\hat{q})$ is well defined for $\hat{q} \in (q_0, \bar{q})$. Lemma 3(b) reveals a monotone correspondence between r and entry probability q^* in any symmetric equilibrium with pure-strategy bidding. The fact of $\hat{q} > q_0$ (Theorem 2(a)) and the definition of r_0 (Definition 4) lead to the following.

Lemma 5 $r(\hat{q}) < r_0$.

When $r(\hat{q})$ falls below the cutoff $\bar{r}$, the optimal contest is straightforward by Theorem 4.

Theorem 6 *Suppose* $\frac{V}{M} < \frac{\alpha}{\alpha-1}\Delta$. *Whenever* $r(\hat{q}) \leq \bar{r}$, *the contest designer can achieve the upper bound* $\bar{x}_T^*$ *by setting* $r = r(\hat{q})$. *This induces a symmetric equilibrium with pure-strategy bidding. Potential bidders enter the contest with a probability* $\hat{q}$ (< 1) *in the symmetric equilibrium.*

Consider a benchmark setting with a fixed number of M participants or free entry. A higher r elicits higher bids whenever a pure-strategy equilibrium prevails, i.e.,

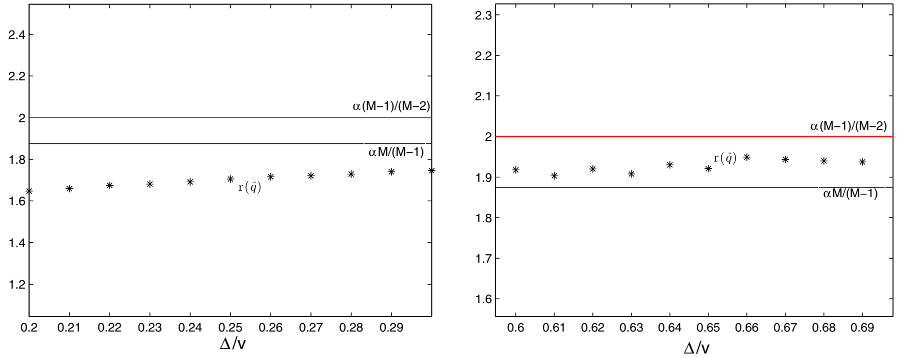

Fig. 1 $r(\hat{q})$ for various $\frac{\Delta}{V}$, given $M = 5, \alpha = 1.5$

whenever $r \le \alpha(1 + \frac{1}{M-1})$.[21] The conventional wisdom in the contest literature holds that $r = \alpha(1 + \frac{1}{M-1})$ would emerge as the optimum. In our setting, the optimal cutoff $\frac{\alpha M}{M-1}(1 - \frac{M\Delta}{V})$ must fall below $\alpha(1 + \frac{1}{M-1})$ when $\frac{V}{M} \ge \frac{\alpha}{\alpha-1}\Delta$. As discussed above, a higher r would lead to random entry and a suboptimal level of overall bids.

When $\frac{V}{M} < \frac{\alpha}{\alpha-1}\Delta$, the optimum $r(\hat{q})$ may either exceed or fall below $\alpha(1 + \frac{1}{M-1})$. Figure 1 provides a sample of our numerical exercises that compares the two settings. Each asterisk corresponds to a numerically computed $r(\hat{q})$ for a given combination of M and $\frac{\Delta}{V}$. The observations in the left panel demonstrate the incidence of $r(\hat{q}) < \alpha(1 + \frac{1}{M-1}) < \bar{r}$: The optimal incentive is softer than that in the benchmark case. In the right panel, the results illustrate the possibility of the opposite: The optimum $r(\hat{q})$ is found in the range $(\alpha(1 + \frac{1}{M-1}), \bar{r})$. However, a higher-powered incentive beyond $r(\hat{q})$ would not pay off: By Theorem 2 and Lemma 3(b), the expected overall bid must strictly decrease when r exceeds $r(\hat{q})$, although it continues to induce pure-strategy bidding, at least before r reaches the cutoff $\bar{r}$.

Further, note from Fig. 1 that $r(\hat{q})$ is not monotonic with Δ for given V. We will show in Sect. 5.3 that $\hat{q}$ monotonically decreases with Δ. Similar patterns prevail when M increases. We omit these graphs to save space.

However, recall that setting r to $r(\hat{q})$ is optimal if and only if it leads to pure-strategy bidding. The optimum becomes less than explicit when $r(\hat{q})$ exceeds the cutoff $\bar{r}$. An equilibrium with pure-strategy bidding is not guaranteed, which casts doubt on the robustness of Theorem 6. As enlightened by Corollary 2, this concern can be dismissed only in contests with small pools of potential participants, i.e., $M = 2, 3$. The condition of $r \le \bar{r}$ may not hold when M gets large. Technical difficulty prevents us from providing a more complete analysis of this scenario. However, recall that $\bar{r}$ is a conservative sufficient condition to guarantee the regularity of the function $\tilde{\pi}(x_i)$. Our numerical exercises find in all observations that a regular payoff function remains whenever $r \in (\bar{r}, r_0]$.

Despite the incomplete analytical account, we show subsequently that the upper bound can always be achieved in a Tullock contest by allowing entry fees/subsidies.

21 In our setting, if entry does not involve a fixed entry cost, all the M potential bidders will participate.

4.2 Contest design by entry fees/subsidies

We now consider entry fees or subsidies as an auxiliary instrument for contest design. The designer is allowed to charge an entry fee F to participating bidders. The entry fee is allowed to be negative, in which case it becomes an entry subsidy. The contest designer commits to a contingent prize V_N, which is a single prize to be awarded to the winner when N entrants show up. The prize purse can be financed by the revenue collected through entry fees. The designer is subject to a budget constraint $V_N \leq V + NF$.

The contest designer chooses a combination of (r, F) to maximize the expected overall bid of the contest. To maximize the overall expected bid, the contest designer must exhaust her original budget V and the revenue collected through entry fees, i.e., $V_N = V + NF$, in the optimum.

This family of contests is a subset of the general mechanisms studied in Sect. 3. It deserves to be pointed out that an entry fee or subsidy can be viewed as a prize (positive or negative) awarded to a participant. The mechanism is equivalent to a multi-prize contest with a contingent prize schedule $(v_1^N, \ldots, v_N^N)$ with $v_1^N = V + (N-1)F$ and $v_i^N = -F$ for all $i \neq 1$. Hence, our general existence theorem does not lose its bite. By Theorem 2, the overall bid in such a contest is also bounded from above by $\bar{x}_T^*$. Hence, a contest mechanism with entry fees/subsidies must reach an optimum if it leads to an equilibrium with an entry probability $\hat{q}$ and pure-strategy bidding: The contest would elicit an expected overall bid of $\bar{x}_T^*$, which cannot be surpassed by other mechanisms under symmetric entry. We show that achieving this upper bound is always feasible with properly set (r, F).

Theorem 7 *Consider a Tullock contest with a contingent prize scheme* $(v_1^N, \ldots, v_N^N)$ *where* $v_1^N = V + (N-1)F$ *and* $v_i^N = -F$ *for all* $i > 1$. N *denotes the number of entrants, which is verifiable. For each given discriminatory power* $r \in (0, \alpha)$*, there exists a unique entry fee/subsidy* $F(r) \in (-\frac{V}{M}, \infty)$*, such that the Tullock contest induces an equilibrium entry probability* $\hat{q}$ *and pure-strategy bidding upon entry. The contest elicits an expected overall bid of* $\bar{x}_T^*$.

When r falls below α, participating bidders have a concave payoff function regardless of entry probability $q \in (0, 1]$, and pure-strategy bidding must result for the above prevailing prize structure. When each $r \in (0, \alpha)$ is bundled with a properly chosen entry fee/subsidy $F(r)$, an equilibrium with entry probability $\hat{q}$ will emerge and leave zero surplus to entrants, which elicits $\bar{x}_T^*$. It should be noted that the optimal combination $(r, F(r))$ is not unique. As implied by our proof, a lower (higher) r can be complemented by a higher (lower) F to induce an equilibrium with entry probability $\hat{q}$.

It should also be noted that the mechanism does not require that the designer observe the exact number of participants. The amount of entry fee/subsidy is not contingent on the actual number of participants. The key for the contest designer is to commit to topping up the prize purse by the entire revenue collected through entry fees/subsidies.

In this section, we allow the contest designer to choose a combination of the discriminatory parameter r and a matching entry fee/subsidy. Finally, we include the following remark, which interprets our results from an alternative perspective.

Remark 1 Theorem 7 establishes that the upper bound can always be achieved by properly chosen combinations of (r, F). This analysis can be interpreted differently: For each exogenously given $r \leq \alpha$, one can always identify a proper entry fee/subsidy $F(r)$, such that $(r, F(r))$ constitutes an optimal mechanism. This implies the sufficiency of using prize structure as the sole instrument to achieve the optimum for given $r \leq \alpha$.

4.3 Implementability of $\bar{x}_T^*$ by Multiple Fixed (Positive) Prizes

Section 4.2 shows that the upper bound $\bar{x}_T^*$ can always be achieved by a nested Tullock contest with a contingent multi-prize scheme. In this section, we investigate to what extent a prize scheme of multiple fixed positive prizes can achieve the same goal.

Let the designer split her total budget V into $T \in \{2, \ldots, M\}$ strictly positive prizes denoted by $(v_1, \ldots, v_T)$, with $\sum_{i=1}^{T} v_i = V$. We obtain the following result.

Theorem 8 *(i) When $\frac{V}{M} < \frac{\alpha}{\alpha-1}\Delta$, a contest with multiple fixed strictly positive prizes can never achieve the upper bound $\bar{x}_T^*$. (ii) When $\frac{V}{M} \geq \frac{\alpha}{\alpha-1}\Delta$, a nested Tullock contest with multiple fixed strictly positive prizes can achieve the upper bound $\bar{x}_T^*$ when $r \in (\frac{\alpha M}{M-1}(1 - \frac{M\Delta}{V}), \frac{\alpha M}{M-1}]$.*

A dedicated proof is unnecessary. When $\frac{V}{M} < \frac{\alpha}{\alpha-1}\Delta$, achieving the upper bound $\bar{x}_T^*$ requires that the contest induce random entry in symmetric equilibrium. Recall the second necessary condition in Sect. 3.4: A total rent V must be given away whenever some bidder(s) enter(s). This condition can never be satisfied under a noncontingent multi-prize schedule. To be more specific, suppose that potential bidders enter with the contest with the same probability q. If the entire budget is concentrated on a single prize, a rent of V is given away with a probability $1 - (1 - q)^{M-1}$, which is the probability that at least one bidder enters. With multiple prizes, however, only v_1 is given away with this probability. A prize $v_t, t > 1$, would not be awarded if fewer than t bidders enter, in which case the rent allocated to the prize is not fully utilized to incentivize bidding. Because of such rent "leakage," a contest with noncontingent multiple positive prizes is suboptimal and can never achieve the upper bound $\bar{x}_T^*$.

By contrast, when $\frac{V}{M} \geq \frac{\alpha}{\alpha-1}\Delta$, the above argument no longer bites. In this case, the optimum requires deterministic entry, i.e., $\hat{q} = 1$. Rent leakage thus would not occur. A multi-prize contest can turn out to be optimal for $r \in (\frac{\alpha M}{M-1}(1 - \frac{M\Delta}{V}), \frac{\alpha M}{M-1}]$ in order to induce all players to enter with $\hat{q} = 1$.

5 Discussion and extensions

In this section, we discuss several remaining issues to expand the scope of our analysis. First, we allow the contest designer to exclude a subset of potential bidders from the competition. We explore the optimal size of the pool of potential bidders when exclusion is possible. Second, we discuss the issue of asymmetric equilibria. Third, the role played by the entry cost Δ is explored in depth.

5.1 Optimal shortlisting

We now allow the contest designer to invite only a subset of the M potential bidders to participate. We assume that only the invited bidders can decide whether to participate in the contest, while they continue to play a symmetric equilibrium. The conventional wisdom holds that a contest elicits higher bids when it involves a larger number of contestants. In what follows, we demonstrate that in our setting, exclusion can improve the efficiency of the contest in our setting.

The expected overall bid in a contest is bounded from above by $\bar{x}_T^*$. The exact amount of $\bar{x}_T^*$, however, depends on the number (M) of potential bidders. Let $\bar{x}_T^*(M)$ be the upper bound of overall bid for a contest with M potential bidders. The function $\bar{x}_T^*(M)$ exhibits the following property.

Lemma 6 *The upper bound $\bar{x}_T^*(M)$ decreases (increases) with M for all M that satisfies $\frac{V}{M} < (\geq)\frac{\alpha}{\alpha-1}\Delta$.*

Lemma 6 yields direct implications for the contest design: A contest is less able to elicit bids if it involves too large a pool of potential bidders. When the contest designer can structure the contest properly to implement the upper bound $\bar{x}_T^*$, she can be strictly better off by excluding potential bidders. Define $\bar{M} \triangleq \min(N| \frac{V}{N} < \frac{\alpha}{\alpha-1}\Delta)$, and assume $\bar{M} < M$ without loss of generality. Lemma 6 means that the upper bound $\bar{x}_T^*(M)$ is essentially single peaked, and it is maximized at either $\bar{M}$ or $\bar{M}-1$. We obtain the following.

Theorem 9 *Suppose $\bar{M} < M$. When the contest designer is allowed to exclude potential bidders, $M^*(= \bar{M}$ or $\bar{M}-1)$ contestants would be invited.*

By inviting $\bar{M}$ or $\bar{M}-1$ of potential bidders and adopting the optimal design discussed in Sects. 4.1 and 4.2, the contest designer elicits an overall bid $\bar{x}_T^*(\bar{M})$ or $\bar{x}_T^*(\bar{M}-1)$, which, by Lemma 6, is unambiguously higher than what she can possibly achieve if she engages the entire pool of M potential bidders.

Our result thus provides an alternative rationale for excluding players in a setting with homogeneous bidders but endogenous entry. The logic resembles that underlying the choice of optimal discriminatory power r. When a contest involves too many potential bidders, each would enter the contest less often and bid less after entry, as they anticipate a more intense competition and expect a smaller share of the rent.

However, it should be noted that in general it is difficult to pin down the exact optimum, $\bar{M}$ vis-à-vis $\bar{M}-1$. The resultant overall bid of the contest varies discretely between the two cases, as the former involves random entry, while the latter induces deterministic entry. The comparison between the two discrete cases depends on various environmental factors. The following examples demonstrate that either can be optimal.

Let K denote the number of invited bidders.

Example 1 Consider a setting with $M = 20$, $V = 10$, $\Delta = 1.1$,and $\alpha = 3$. The $\bar{M}$ is 7. When $K = 7$, the optimal contest elicits a maximal amount of overall bid 4.966 with $\hat{q} = 0.866$. When $K = 6$, the optimal contest elicits a maximal amount of overall bid $\bar{x}_T^*(6) = 4.965$ with $\hat{q} = 1$. Thus the optimum is achieved by inviting $M^* = \bar{M} = 7$ potential bidders.

In this example, the contest designer prefers to induce an equilibrium with *random entry* and does not further limit the pool of the potential bidders below the cutoff $\bar{M}$.

Example 2 Consider a setting with $M = 20$, $V = 10$, $\Delta = 1.1$ and $\alpha = 1.5$. The $\bar{M}$ is 4. When $K = 4$, the optimal contest elicits a maximal amount of overall bid 5.11 with $\hat{q} = 0.797$. When $K = 3$, the optimal contest elicits a maximal amount of overall bid $\bar{x}_T^*(3) = 5.13$ with $\hat{q} = 1$.Thus the optimum is achieved by inviting $M^* = \bar{M} - 1 = 3$ potential bidders.

Example 2 shows the possibility that the contest designer prefers to induce an equilibrium with *deterministic entry* and further limits the pool to $\bar{M} - 1$ potential bidders. Despite the ambiguity, the following is unequivocal in the case of linear bidding cost.[22]

Corollary 4 *When the bidding cost function is linear, i.e.,* $\alpha = 1$, $\bar{M} = 2$, *and the optimal contest invites exactly* $\bar{M} = 2$ *potential bidders.*

5.2 Asymmetric entry

Our analysis has so far focused on symmetric equilibria, in which potential bidders enter the contest with the same probability. As mentioned previously, asymmetric equilibria must arise in this game. One natural and intuitive candidate is the "semi-symmetric" equilibrium, in which a subset of potential bidders enter the contest with the same positive probability and play the same bidding strategy upon entry, while the rest of them stay inactive.

Our results on symmetric equilibria could directly speak to these asymmetric equilibria in general. Let $M'(\leq M)$ be the number of active bidders involved in such a semi-symmetric equilibrium, where "active bidders" are those who enter with a positive probability. Theorem 2 immediately leads to the following corollary.

Corollary 5 *Suppose that a semi-symmetric equilibrium with* $M'(\leq M)$ *active bidders exists. In such an equilibrium, the expected overall bid of the contest must be bounded from above by* $\bar{x}_T^*(M')$*, the upper bound for a hypothetical contest with* M' *potential bidders when they play any symmetric equilibrium.*

Consider an arbitrary M-player contest. Corollary 5 implies that any semi-symmetric equilibrium of the contest cannot strictly outperform the optimum under symmetric entry achieved through optimal shortlisting, i.e., $x_T^*(M^*)$ with $M^* = \bar{M}$ or $\bar{M} - 1$. The remaining question thus boils down to whether the optimum $x_T^*(M^*)$

[22] This corollary is true, since effort elicited would be zero when only one player is invited.

can indeed be implemented in a semi-symmetric equilibrium. The following theorem answers this question.

Theorem 10 *The optimal level of expected overall bid $x_T^*(M^*)$, which is induced under symmetric entry through optimal shortlisting, can also be generated in a semi-symmetric equilibrium involving M^* active players in a properly designed contest with M potential bidders.*

Theorem 10 is straightforward. Consider a contest with M potential bidders. Recall our discussion in Sect. 4. When they play symmetric equilibrium, the optimum $x_T^*(M^*)$ can always be implemented by the contest designer by inviting M^* potential bidders to participate and imposing a set of properly designed contest rules, by either adjusting r or supplementing it with appropriate entry fees/subsidies. Let $\Phi(M^*)$ denote this optimal contest mechanism. In the optimum, the M^* invited bidders enter with a probability $\hat{q}(M^*)$ and place their bids in pure strategy upon entry. The same objective can always be achieved if the contest designer can effectively coordinate on asymmetric equilibria. To see that, suppose that the designer imposes the same contest mechanism $\Phi(M^*)$.[23] Then a semi-symmetric equilibrium must emerge, in which exactly M^* play the same strategy as the aforementioned M^* invited bidders would do in a symmetric equilibrium after shortlisting. They also enter with a probability $\hat{q}(M^*)$ and play the same bidding strategy. This strategy profile must constitute an equilibrium: The M^* potential bidders end up with zero payoff, so the remaining bidders have no incentive to enter.[24]

In summary, in an M-bidder contest, the optimum $\bar{x}_T^*(M^*)$ can be achieved through either a symmetric equilibrium by shortlisting M^* potential bidders, or an asymmetric (semi-symmetric) equilibrium with M^* active bidder. Shortlisting, however, might remain a preferred device: The symmetric equilibrium can arguably be viewed as a natural focal point of a symmetric game, which requires less coordination between players, while more coordination among players is required to support an asymmetric equilibrium among symmetric players. This issue has been discussed previously following Theorem 1 in Sect. 3.3.

5.3 Changes in entry cost Δ

We now explore how a change in entry cost Δ would affect optimal contest design.[25] From (6), $\bar{x}_T(q)$ decreases with Δ regardless of q. Hence, the upper bound $\bar{x}_T^*$ must decrease with Δ for every given M. This implies that a larger entry cost is never desirable to the contest designer, because it unambiguously weakens the potential to elicit bids.

[23] More precisely, the prize allocation scheme has to accomodate the possibility that more than M^* players enter. To rule out such a possibility, we assume that in this case all entrants will be penalized by a severe negative prize.

[24] Detailed proofs are available from authors upon request.

[25] The entry fee/subsidy F in Sect. 4.2 plays a different role from the fixed sunk entry cost Δ: The contest collects F from participants and commits to redistributing the revenue by topping up the prize purse, while the entry cost Δ is purely sunk.

We further consider its impact on the optimal level of entry probability $\hat{q}$. By the proof of Theorem 2, $\hat{q}$ is determined by the properties of the function

$$\xi(q,\Delta) = (\alpha-1)\left\{[1-(1-q)^M]V - Mq\Delta\right\} + Mq[(1-q)^{M-1}V - \Delta].$$

Theorem 2 implies (1) that $\hat{q} \in (q_0, 1)$ if $\frac{V}{M} < \frac{\alpha}{\alpha-1}\Delta$, and $\hat{q} = 1$ if $\frac{V}{M} \geq \frac{\alpha}{\alpha-1}\Delta$, and (2) that in the former case, $\hat{q}$ is determined by $\xi(\hat{q},\Delta) = 0$. We next show that $\hat{q}$ drops with Δ. For this purpose, it suffices to show that the solution $\hat{q}$ of $\xi(\hat{q},\Delta) = 0$ drops with Δ. The proof of Theorem 2 indicates that $\left.\frac{\partial\xi(q,\Delta)}{\partial q}\right|_{q=\hat{q}}$ is strictly negative. Further, $\frac{\partial\xi(q,\Delta)}{\partial\Delta}$ is obviously negative. As a result, $\frac{d\hat{q}}{d\Delta} < 0$ must hold. To put it intuitively, a larger entry cost causes additional rent dissipation, and therefore the optimal contest must limit entry to avoid excessive waste.

Moreover, it is obvious to observe that $\bar{M}$ decreases with Δ as well. Recall from Sect. 5.1 that the optimal contest shortlists $M^* = \bar{M}$ or $\bar{M} - 1$ potential bidders. A decreasing $\bar{M}$ implies that a larger Δ tends to lower the optimal number of shortlisted players as well.[26] The following theorem collects the above observations.

Theorem 11 *$\hat{q}$, $\bar{x}_T^*$ and $\bar{M}$ all drop with entry cost Δ.*

It remains to explore how a larger Δ would affect the required discriminatory parameter $r(\hat{q})$ defined in Sect. 4.1 to achieve the upper bound $\bar{x}_T^*$. In the case of $\frac{V}{M} \geq \frac{\alpha}{\alpha-1}\Delta$ and thus $\hat{q} = 1$, $r(\hat{q})$ drops with Δ because $r(\hat{q}) = \frac{\alpha M}{M-1}(1 - \frac{M\Delta}{V})$ by Theorem 5. However, when $\frac{V}{M} < \frac{\alpha}{\alpha-1}\Delta$ and thus $\hat{q} < 1$, the results of Fig. 1 in Sect. 4.1 show that $r(\hat{q})$ does not change monotonically with Δ. Two effects come into play when Δ increases. It first exercises a direct effect. More costly entry requires that the contest leave more rent to participants for a given entry probability, thereby demanding a less intense competition in the contest, i.e., a smaller r. An indirect effect also emerges. As stated above, more costly entry requires a more limited entry pattern in the optimum, i.e., a smaller $\hat{q}$. Further recall the inverse relation between r and q^*, as stated by Lemma 3. This, in turn, forces r to increase in response to the smaller $\hat{q}$. The two effects conflict with each other, which leads to the mixed observations in Fig. 1.

6 Concluding remarks

Our study is one of the first to investigate the subtle and rich strategic interactions that occur in contests with endogenous entry. We establish the existence of a symmetric equilibrium for the well adopted family of generalized nested lottery contests with contingent prize allocation schemes, and provide sufficient conditions under which entrants engage in pure-strategy bidding in Tullock contests with a single winner prize. Based on these equilibrium analyses, we identify optimal contest designs that

[26] A simulation confirms this.

maximize total expected effort. Our results illustrate the unique and important impacts of entry costs on equilibrium analysis and contest design.

In our analysis, we assume that the actual number of participants is unobservable to participating bidders. One natural question is whether the contest designer could improve the contest design by disclosing the actual number of participating bidders when she is able to observe it. It deserves to be noted that the current setting causes no loss of generality. The upper bound $\bar{x}_T^*$ established in Sect. 3.4 is independent of prevailing contest rules, and it applies to contests with the actual number of participants being disclosed.[27] Hence, whenever a mechanism can successfully achieve the upper bound x_T^*, it must be optimal and (at least weakly) dominate all other possible mechanisms. Hence, disclosure would not improve bidding efficiency further.

Also, our setting (characterized by a common entry cost and simultaneous entry) is only one way to model contests that involve endogenous entry. Other examples include the settings of Kaplan and Sela (2010), who consider all-pay auctions with privately known entry costs, and Fu and Lu (2010), who assume that contestants enter sequentially. Various questions about the optimal design of contests in these diverse settings remain open, and they merit serious consideration in the future.

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Appendix

Proof of Theorem 1 **Part (a) Existence of symmetric equilibrium:** We will start by considering the following auxiliary game, which differs from the original game in terms of action space of players. The payoff structure of the auxiliary game duplicates that of the original game considered in Sect. 3.1. We will establish that there is a symmetric equilibrium for the auxiliary game. In addition, we show that it is also a symmetric equilibrium of the original game.

The auxiliary game is defined as below. There are M contestants who simultaneously choose their two-dimensional actions, which are denoted by $a_i = (a_{i1}, a_{i2}) = (q_i, x_i) \in A$, $i = 1, 2, ..., M$, where the uniform action space $A = [0, 1] \times [0, V^{1/\alpha}]$ is nonempty, convex and compact.[28] Let $\mathbf{x} = (x_1, x_2, ..., x_i, ..., x_N)$ denote the effort profile.

Let $\mathbf{k} = (k_1, k_2, ..., k_i, ..., k_N)$ denote the entry status of players, where k_i is either 0 or 1. k_i denotes the entry status of players with $k_i = 1$ for entry of player i, and

[27] The arguments are similar to that of Sect. 3.4.

[28] This action space A is not for the original game of Sect. 3.1, but for the auxiliary game.

$k_i = 0$ for non-entry of player i. Let K to be the set of all possible $\mathbf{k}$. Similarly, we can define $\mathbf{k}_{-i}$ and K_{-i}, $i = 1, 2, ..., M$. Let $N(\mathbf{k}_{-i}) = 1 + \sum_{j \neq i} k_j$, which is the number of all entrants when i enters.

Given action profile $\mathbf{a} = \{a_1, a_2, ..., a_M\}$ of the M players, the payoff of player i is defined as

$$U_i(\mathbf{a}) = q_i \left\{ \sum_{\mathbf{k}_{-i} \in K_{-i}} \left[\left(\prod_{j \neq i} q_j^{k_j} (1-q_j)^{1-k_j} \right) \left(\sum_{n=1}^{N(\mathbf{k}_{-i})} p_i(n|\mathbf{k}_{-i}, \mathbf{x}) v^n_{N(\mathbf{k}_{-i})} \right) \right] - x_i^\alpha - \Delta \right\}, \quad i = 1, 2, ..., M,$$

where $p_i(n|\mathbf{k}_{-i}, \mathbf{x})$ denotes the probability that entrant i wins the n-th highest prize among prizes $\{v^n_{N(\mathbf{k}_{-i})}\}_{n=1}^{N(\mathbf{k}_{-i})}$ when player $j(\neq i)$ enters if and only if $k_j = 1$. $p_i(n|\mathbf{k}_{-i}, \mathbf{x})$ is determined by the effort profile of player i and other players with $k_j = 1$ in $\mathbf{k}_{-i}$ and the prize allocation rule that has been specified in Sect. 3.1.

Note that this game is a symmetric game as defined by Dasgupta and Maskin (1986) in their Definition 7. We will apply their Theorem 6* in Appendix to establish the existence of symmetric equilibrium in mixed strategy.

In what follows, we show that for each i, the discontinuities of U_i are confined to a subset of a continuous manifold of dimension less than M as required by page 7 of Dasgupta and Maskin (1986). Following the notations on page 22 of Dasgupta and Maskin (1986). Let $Q = \{2\}$, $D(i) = 1$, and f^1_{ij} to be an identity function. Following their $(A1)$ of page 22, we define manifold $A^*(i) = \{\mathbf{a} \in A | \exists j \neq i, \exists k \in Q, \exists d, 1 \leq d \leq D(i)$ such that $a_{jk} = f^d_{ij}(a_{ik})\}$. Clearly, $A^*(i)$ is of dimension less than M. Let $A^{**}(i)$ denote the set of discontinuous points of $U_i(\mathbf{a})$. $\forall \mathbf{a} \in A^{**}(i)$, we must have that $q_i > 0, x_i = 0$ and $\exists j \neq i$, such that $q_j > 0$ and $x_j = 0$. Clearly, $U_i(\mathbf{a})$ $A^{**}(i) \subset A^*(i)$, since any element in $A^{**}(i)$ must satisfy the following conditions: For $k = 2 \in Q$, $\exists j \neq i$, such that $x_j = f^1_{ij}(x_i)$, i.e., $a_{j2} = f^1_{ij}(a_{i2})$.

According to their Theorem 6*, we need to verify the following conditions hold.

First, as constructed above, $U_i(\mathbf{a})$ is continuous except on a subset $A^{**}(i)$ of $A^*(i)$, where $A^*(i)$ is defined by $(A1)$.

Second, clearly, we have $\sum_i U_i(\mathbf{a}) = \sum_{\mathbf{k} \in K}[(\prod_j q_j^{k_j}(1 - q_j)^{1-k_j})(\sum_{n=1}^{N(\mathbf{k})} v^n_{N(\mathbf{k}_{-i})})] - \sum_i q_i(x_i^\alpha + \Delta)$, which is continuous and thus upper semi-continuous.

Third, $U_i(\mathbf{a})$ clearly is bounded on $A = [0, 1] \times [0, v^{1/\alpha}]$.

Fourth, we verify that Property (α^*) of page 24 is satisfied. Define B^2 as the unit circle with the origin as its center, i.e., $B^2 = \{\mathbf{e} = (q, x) \mid q^2 + x^2 = 1\}$. Pick up any continuous density function $v(\cdot)$ on B^2 such that $v(\mathbf{e}) = 0$ iff $e_1 \leq 0$ or $e_2 \leq 0$. Note that $U_i(a_i, \mathbf{a}_{-i})$ is continuous in a_{i1} and lower semi-continuous in a_{i2}. $\forall \mathbf{a} = (\bar{a}_i, \mathbf{a}_{-i}) \in A^{**}(i)$, clearly we have that for any $\mathbf{e}$ such that $v(\mathbf{e}) > 0$ (i.e., $\min(e_1, e_2) > 0$), $\liminf_{\theta \to 0^+} U_i(\bar{a}_i + \theta \mathbf{e}, \mathbf{a}_{-i}) > U_i(\bar{a}_i, \mathbf{a}_{-i})$ as $\theta > 0, e_2 > 0$ and $q_i > 0, x_i = 0$ in $\bar{a}_i$. This property holds as we have the following result: $\forall \mathbf{a} \in A^{**}(i)$, fix $\mathbf{a}_{-i}$ and $q_i(> 0)$ in $\mathbf{a}$, $U_i(\mathbf{a})$ is discontinuous at $x_i = 0$. In other words, when x_i increases from 0 to an $\varepsilon > 0$, $U_i(\mathbf{a})$ has a jump upward. This leads to that $\int_{B^2}[\liminf_{\theta \to 0^+} U_i(\bar{a}_i + \theta \mathbf{e}, \mathbf{a}_{-i}) v(\mathbf{e}) d\mathbf{e}] > U_i(\bar{a}_i, \mathbf{a}_{-i}), \forall \bar{a}_i \in A_i^{**}(i), \mathbf{a}_{-i} \in$

$A^{**}_{-i}(\bar{a}_i)$, where $A^{**}_i(i)$ is the collection of all $\bar{a}_i$ of player i that appear in $A^{**}(i)$, $A^{**}_{-i}(\bar{a}_i)$ is the collection of others' actions $\mathbf{a}_{-i}$ such that $\mathbf{a} = (\bar{a}_i, \mathbf{a}_{-i}) \in A^{**}(i)$. This confirms that Property (α^*) holds for the above game.

Thus according to Theorem 6* of Dasgupta and Maskin (1986), there exists a symmetric mixed strategy equilibrium. Without loss of generality, we use $\mu_1(q)$ to denote the equilibrium probability measure of action q, and use $\mu_2(x)$ to denote the equilibrium probability measure of action x.

Next we show that for any strategy profile of players $\{(\mu_{i1}(q_i), \mu_{i2}(x_i))\}$. The players' payoffs are same from strategy profile of players that is defined as $\{(E_{\mu_{i1}}q_i, \mu_{i2}(x_i))\}$. The expected utility of player i from profile $\{(\mu_{i1}(q_i), \mu_{i2}(x_i))\}$ is

$$
\begin{aligned}
&EU_i(\mathbf{a})\\
&= E_{q_i} E_{\mathbf{q}_{-i}} E_{\mathbf{x}} \left\{ q_i \left[\sum_{\mathbf{k}_{-i} \in K_{-i}} \left(\prod_{j \neq i} q_j^{k_j} (1-q_j)^{1-k_j} \right) \left(\sum_{n=1}^{N(\mathbf{k}_{-i})} p_i(n|\mathbf{k}_{-i}, \mathbf{x}) v^n_{N(\mathbf{k}_{-i})} \right) \right] - x_i^{\alpha} - \Delta] \right\}\\
&= [Eq_i] E_{\mathbf{x}} E_{\mathbf{q}_{-i}} \left[\sum_{\mathbf{k}_{-i} \in K_{-i}} \left(\prod_{j \neq i} q_j^{k_j} (1-q_j)^{1-k_j} \right) \left(\sum_{n=1}^{N(\mathbf{k}_{-i})} p_i(n|\mathbf{k}_{-i}, \mathbf{x}) v^n_{N(\mathbf{k}_{-i})} \right) \right] - x_i^{\alpha} - \Delta]\\
&= [Eq_i] E_{\mathbf{x}} \left[\sum_{\mathbf{k}_{-i} \in K_{-i}} \left(\prod_{j \neq i} (Eq_j)^{k_j} (1-Eq_j)^{1-k_j} \right) \left(\sum_{n=1}^{N(\mathbf{k}_{-i})} p_i(n|\mathbf{k}_{-i}, \mathbf{x}) v^n_{N(\mathbf{k}_{-i})} \right) \right] - x_i^{\alpha} - \Delta], \quad \forall i.
\end{aligned}
$$

The above result means that given others take strategy $(E_{\mu_1}q, \mu_2(x))$, the same strategy is also the best strategy for player i. Otherwise, $(\mu_1(q), \mu_2(x))$ would not be the optimal strategy for player i when others take the same strategy $(\mu_1(q), \mu_2(x))$. Therefore, $(E_{\mu_1}q, \mu_2(x))$ is a *symmetric equilibrium* for the above game.

It is easy to see that $(q^*, \mu^*(x)) = (E_{\mu_1}q, \mu_2(x))$ is a symmetric equilibrium for our original game based on the way the auxiliary game is constructed. $U_i(\mathbf{a})$ equals player i's expected payoffs when he enters with probability q_i and exerts effort x_i when he enters, given that other bidder j enters with probability q_j and exerts effort x_j when he enters. This claim also holds when they adopt any other entry strategies with measure $\{\mu_{i1}(q), i = 1, 2, ..., M\}$ due to (11). According to (11), only the expected entry probabilities $\{E_{\mu_{i1}}q, i = 1, 2, ..., M\}$ count.

We must have $q^* = E_{\mu_1}q \in (0, 1]$ since $q^* = E_{\mu_1}q = 0$ cannot be an entry equilibrium when $\Delta < V$ (Assumption 1).

Part (b): The equilibrium payoff cannot be negative. When $q^* = E_{\mu_1}q \in (0, 1)$, we must have the equilibrium payoffs of player to be zero as otherwise it cannot be an equilibrium as the player would enter with probability 1 and earn a positive payoff. □

Proof of Lemma 1 Let $f_1(q) = [1-(1-q)^M]v - Mq\Delta$, and $f_2(q) = (1-q)^{M-1}v - \Delta$. Without loss of generality, we focus on the case where $\frac{V}{M} < \Delta$ such that $\bar{q} \in (0, 1)$ is defined as $f_1(\bar{q}) = 0$. The first order derivative of $f_1(q)$ is $f_1'(q) = Mf_2(q)$, which is a decreasing function of q. $f_1'(q)$ is positive when $q = 0$, and it is negative when $q = 1$.

q_0 is defined as $f_2(q_0) = 0$. Therefore, $f_1(q)$ increases on $[0, q_0]$, and decreases from $[q_0, 1)$. $f_1(q)$ thus has two zero points, i.e., $\{0, \bar{q}\}$, and $q_0 < \bar{q}$. □

Proof of Theorem 2 Define an increasing transformation of $\bar{x}_T(q)$:

$$\Psi(q) = [\bar{x}_T(q)]^{\alpha} = (Mq)^{\alpha-1}\left\{[1-(1-q)^M]V - Mq\Delta\right\}$$

Note that $\Psi(q)|_{q=0} = 0$. We have

$$\frac{d\Psi(q)}{dq} = f(q)\,q^{\alpha-2}M^{\alpha-1},$$

where

$$f(q) = (\alpha-1)\underbrace{\left\{[1-(1-q)^M]V - Mq\Delta\right\}}_{f_1(q)} + Mq\underbrace{[(1-q)^{M-1}V - \Delta]}_{f_2(q)}.$$

We have

$$f'(q) = MV(1-q)^{M-2}[\alpha-(M+\alpha-1)q] - \alpha M\Delta.$$

Note that $f'(0) = \alpha MV - \alpha M\Delta > 0$, $f'(1) = -\alpha M\Delta < 0$ and $f'(q)$ decreases with $q \in (0, \frac{\alpha}{M+\alpha-1}]$. Clearly, $f'(q) < 0$ when $q \in [\frac{\alpha}{M+\alpha-1}, 1]$. Then there exists a unique $q_c \in (0, \frac{\alpha}{M+\alpha-1})$, such that $f'(q_c) = 0$, which means q_c is the maximum point of $f(q)$.

(i) Since $f(0) = 0$, $f(q_c) > 0$ and $f(1) = (\alpha-1)V - \alpha M\Delta = \alpha(V - M\Delta) - V < 0$ when $\frac{\alpha-1}{\alpha}\frac{V}{M} < \Delta$, then there must exist a unique $\hat{q} \in (q_c, 1)$, such that $f(\hat{q}) = 0$. Note that $f'(q) < 0$ on $(q_c, 1)$. Clearly, $f(q) > 0$ when $0 < q < \hat{q}$; and $f(q) < 0$ when $\hat{q} < q < 1$.

Since $\frac{d\Psi(q)}{dq}$ shares the same sign with $f(q)$, we have that $\frac{d\Psi(q)}{dq} > 0$ when $0 < q < \hat{q}$; and $\frac{d\Psi(q)}{dq} < 0$ when $\hat{q} < q < 1$. This implies $\hat{q} = \arg\max_q \Psi(q)$, i.e., $\hat{q} = \arg\max_q \bar{x}_T(q)$.

Let $f_1(q) = [1-(1-q)^M]V - Mq\Delta$, and $f_2(q) = (1-q)^{M-1}V - \Delta$. The first order derivative of $f_1(q)$ is $f_1'(q) = Mf_2(q)$, which is a decreasing function of q. $f_1'(q)$ is positive when $q = 0$, and it is negative when $q = 1$. $f_2(q)$ decreases with q, and q_0 is defined as $f_2(q_0) = 0$. Therefore, $f_1(q)$ increases on $[0, q_0]$, and decreases from $[q_0, 1)$. $f_1(q)$ thus must be nonnegative on $[0, q_0]$. Thus, both $f_1(q)$ and $f_2(q)$ are positive for $q \in (0, q_0]$. Thus a zero point $(\hat{q})$ of $f(q)$ must fall in $(q_0, 1)$.

(ii) Since $f(0) = 0$, $f(q_c) > 0$ and $f(1) = (\alpha-1)V - \alpha M\Delta = \alpha(V - M\Delta) - V \geq 0$ when $\frac{\alpha-1}{\alpha}\frac{V}{M} \geq \Delta$. Then $f(q) \geq 0$ when $0 < q \leq 1$. Since $\frac{d\Psi(q)}{dq}$ shares the same sign with $f(q)$, we have that $\frac{d\Psi(q)}{dq} \geq 0$ when $0 < q \leq 1$. This implies $\arg\max_q \Psi(q) = 1$, i.e., $\hat{q} = 1$.

The single-peakedness of $\bar{x}_T(q)$ is clear based on the property of $f(q)$ illustrated above.

$\bar{q} < 1$ if and only if $\frac{V}{M} < \Delta$. We can focus on this case to show $\hat{q} < \bar{q}$. By the proof of Lemma 1, we know in this case both $f_1(q)$ and $f_2(q)$ are positive when $q \in [0, q_0]$ and both are negative when $q > \bar{q}$. Thus the zero point ($\hat{q}$) of $f(q)$ must fall in $[q_0, \bar{q}]$.

We next prove the monotonicity of $\hat{q}$ with respect to M. For this purpose, we only need to show this holds when M is high such that $\hat{q} < 1$. Note that in this case $\hat{q}$ is defined by $f(q) = 0$.

We have shown above that $f'(q)|_{q=\hat{q}} < 0$ when $\hat{q} < 1$. Further note that

$$\frac{\partial f(q)}{\partial M}|_{q=\hat{q}} = -(\alpha - 1)\left[(1-\hat{q})^M \ln\left(1-\hat{q}\right) V + \hat{q}\Delta\right] \\ +\hat{q}\left[(1-\hat{q})^{M-1}V - \Delta\right] + M\hat{q}(1-\hat{q})^{M-1}\ln\left(1-\hat{q}\right) V.$$

When $q = \hat{q}$, the last two terms are negative since $\hat{q} \in (q_0, 1)$ and $f_2(q_0) = 0$. The first term has an opposite sign with the following expression:

$$(1-\hat{q})^M \ln\left(1-\hat{q}\right) V + \hat{q}\Delta \\ > (1-\hat{q})^M \ln\left(1-\hat{q}\right) V + \hat{q}(1-\hat{q})^{M-1}V \\ = \left[(1-\hat{q})\ln\left(1-\hat{q}\right) + \hat{q}\right] \cdot (1-\hat{q})^{M-1}V.$$

The first inequality holds since $(1-\hat{q})^{M-1}V - \Delta < 0$. Define $g(q) = (1-q)\ln(1-q) + q, q \in (0, 1)$. Since $g'(q) = -\ln(1-q) > 0$, $g(0) = 0$. Then $g(\hat{q}) > 0$. Therefore, the first term is negative at $q = \hat{q}$ as well.

Since $\frac{\partial f(q)}{\partial q}|_{q=\hat{q}} < 0$ and $\frac{\partial f(q)}{\partial M}|_{q=\hat{q}} < 0$, these gives $\hat{q}$ decreases with M. □

Proof of Lemma 2 Consider an arbitrary potential bidder i who has entered the contest. Suppose that all other potential bidders play a strategy (q, x) with $x > 0$.[29] He chooses his bid x_i to maximize his expected payoff

$$\pi_i(x_i | q, x) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}\left[\frac{x_i^r}{x_i^r + (N-1)x^r}V - x_i^\alpha\right].$$

Differentiating $\pi_i(x_i | q, x)$ with respect to x_i yields

$$\frac{d\pi_i(x_i | q, x)}{dx_i} = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}\frac{(N-1)rx_i^{r-1}x^r V}{[x_i^r + (N-1)x^r]^2} - \alpha x_i^{\alpha-1}.$$

Suppose that a symmetric equilibrium with pure-strategy bidding exists. The (pure) bidding strategy in the equilibrium can be solved by the first order condition $\frac{d\pi_i}{dx_i}|_{x_i=x} =$

[29] It is impossible to have all participating bidders bid zero deterministically in an equilibrium. When all others bid zero, a participating bidder would prefer to place an infinitely small positive bid, which allows him to win the prize with probability one.

0 given the equilibrium entry probability q^*, while q^* is characterized by the zero-payoff condition of mixed-strategy equilibria.[30]

According to the first order condition $\frac{d\pi_i(x_i)}{dx_i} = 0$ and the symmetry condition $x_i = x,\ x^*$ must solve

$$\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} \frac{(N-1)rV}{N^2 x^*} - \alpha x^{*\alpha-1} = 0,$$

which yields

$$x^*(q) = \left[\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} \frac{N-1}{N^2}\frac{rV}{\alpha}\right]^{\frac{1}{\alpha}}.$$

The equilibrium expected payoff is

$$\begin{aligned}\pi^*(x^*(q), q) &= \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} \frac{V}{N} \\ &\quad - \left[\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} \frac{N-1}{N^2}\frac{rV}{\alpha}\right] \\ &= \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} \frac{V}{N}\left(1 - \frac{N-1}{N}\frac{r}{\alpha}\right).\end{aligned}$$

By entering the contest and submit the bid $x^*(q)$, every potential contestant i ends up with an expected payoff

$$\pi^*(x^*(q), q) - \Delta.$$

By Theorem 1 (b), each potential bidder receives a zero expected payoff for the equilibrium entry q^*, i.e., $\pi^*(x^*(q^*), q^*) = \Delta$.

The expected overall effort of the contest (x_T^*) obtains as

$$x_T^* = Mq^* x^*(q^*) = Mq^* \left[\sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N} \frac{N-1}{N^2}\frac{rV}{\alpha}\right]^{\frac{1}{\alpha}}.$$

□

Proof of Lemma 3 By Lemma 2, q^* satisfies $F(q^*, r) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1 - q^*)^{M-N}\frac{V}{N}(1 - \frac{N-1}{N}\frac{r}{\alpha}) - \Delta = 0$. Apparently, $F(q^*, r)$ is continuous in and differentiable with both arguments. We first claim that $F(q^*, r)$ strictly decreases with q^*.

[30] Note that this two-step procedure is only valid when the contest induces pure-strategy bidding in the equilibrium. As will be shown later, such an equilibrium would not exist if r is sufficiently high.

Define $\pi_N = \frac{V}{N}(1 - \frac{N-1}{N}\frac{r}{\alpha})$. Taking its first order derivative yields

$$\begin{aligned}\frac{F(q^*, r)}{dq^*} &= \sum_{N=1}^{M} C_{M-1}^{N-1}[(N-1)q^{*N-2}(1-q^*)^{M-N} \\ &\quad -(M-N)q^{*N-1}(1-q^*)^{M-N-1}]\pi_N \\ &= \sum_{N=1}^{M} C_{M-1}^{N-1}(N-1)q^{*N-2}(1-q^*)^{M-N}\pi_N \\ &\quad - \sum_{N=1}^{M} C_{M-1}^{N-1}(M-N)q^{*N-1}(1-q^*)^{M-N-1}\pi_N \\ &= (M-1)\left\{\sum_{N=2}^{M} C_{M-2}^{N-2}q^{*N-2}(1-q^*)^{M-N}\pi_N \right. \\ &\quad \left. - \sum_{N=1}^{M-1} C_{M-2}^{N-1}q^{*N-1}(1-q^*)^{M-N-1}\pi_N\right\} \\ &= (M-1)\sum_{N=1}^{M-1} C_{M-2}^{N-1}q^{*N-1}(1-q^*)^{M-N-1}(\pi_{N+1} - \pi_N),\end{aligned}$$

which is obviously negative because $\pi_N = \frac{1}{N}\left[1 - \left(1 - \frac{1}{N}\right)\frac{r}{\alpha}\right]V \geq 0$ and it monotonically decreases with N.

When all other potential contestants play $q = 0$, a potential contestant receives a payoff $V - \Delta > 0$, and he must enter with probability one. When all others play $q = 1$, a participating contestant receives negative expected payoff since $\frac{V}{M} < \Delta$, which cannot constitute an equilibrium either. Hence, a unique $q^* \in (0, 1)$ must exist that solves $\pi^*(x^*, q) = \Delta$. Each potential contestant is indifferent between entering and staying inactive when all others play the strategy. This constitutes an equilibrium.

Moreover, $F(q^*, r)$ strictly decreases with r. Since it also strictly decreases with q^*, the part (b) of the lemma is then verified. □

Proof of Lemma 4 Denote $k_i = x_i^\alpha$, $k^* = x^{*\alpha}$, $t = \frac{r}{\alpha} \in (0, \overline{r}]$, then $\tilde{\pi}_i(x_i)$ can be rewritten as

$$\tilde{\pi}_i(k_i) = \sum_{N=1}^{M} C_{M-1}^{N-1}q^{*N-1}(1-q^*)^{M-N}\frac{k_i^t}{k_i^t + (N-1)k^{*t}}V - k_i,$$

Differentiating $\tilde{\pi}_i$ with respect to k_i yields

$$\frac{d\tilde{\pi}_i}{dk_i} = \sum_{N=1}^{M} C_{M-1}^{N-1}q^{*N-1}(1-q^*)^{M-N}\frac{(N-1)tk_i^{t-1}k^{*t}V}{[k_i^t + (N-1)k^{*t}]^2} - 1.$$

Note

$$k^* = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}\frac{N-1}{N^2}tV.$$

To verify that k^* is the global maximizer of $\tilde{\pi}_i(k_i)$ given that all other participants exert the same effort. Define $p_i(k_i, \mathbf{k}_{-i}; N) = \frac{k_i^t}{k_i^t+(N-1)k^{*t}}$. One can verify $\xi_N(k_i) = \left.\frac{\partial^2 p_i(k_i,\mathbf{k}_{-i};N)}{\partial k_i^2}\right|_{k_{-i}=k^*} = \frac{-(t+1)k_i^t+(t-1)(N-1)k^{*t}}{[k_i^t+(N-1)k^{*t}]^3} t k_i^{t-2}(N-1)k^{*t}$. It implies that $\Phi_N(k_i) = \left.\frac{\partial p_i(k_i,\mathbf{k}_{-i};N)}{\partial k_i}\right|_{k_{-i}=k^*}$ is not monotonic: It is positive if $k_i^t < \frac{t-1}{t+1}(N-1)k^{*t}$, and negative if $k_i^t > \frac{t-1}{t+1}(N-1)k^{*t}$. Clearly $\frac{t-1}{t+1}(N-1) \le 1$ if and only if $t \le \frac{N}{N-2}$. Because $t \le \min\{1+\frac{1}{M-2}, 2\}$, we must have $\frac{t-1}{t+1}(N-1) < 1$ for all $N \le M$.

Let $\Phi(k_i) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}\frac{\partial p_i(k_i,\mathbf{k}_{-i};N)}{\partial k_i}|_{k_{-i}=k^*}$, and $\xi(k_i) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}\frac{\partial^2 p_i(k_i,\mathbf{k}_{-i};N)}{\partial k_i^2}|_{k_{-i}=k^*}$. The above results imply that $k_i^t > \frac{t-1}{t+1}(N-1)k^{*t}$ when $k_i = k^*$ for all $N \le M$, which means that $\xi(k_i)|_{k_i=k^*} < 0$. This leads to that $\left.\frac{d^2\tilde{\pi}_i(k_i)}{dk_i^2}\right|_{k_i=\mathbf{k}_{-i}=k^*} = V\,\xi(k_i)|_{k_i=k^*} < 0$. Hence, $k_i = k^*$ must be at least a local maximizer of when $k_{-i} = k^*$.

Since when $k_i < [\frac{t-1}{t+1}]^{1/t}k^*$, $\xi_N(k_i) > 0$ for all $N \le M$, we have $\xi(k_i) > 0$ when $k_i < [\frac{t-1}{t+1}]^{1/t}k^*$, which means that $\Phi(k_i)$ increases when $k_i < [\frac{t-1}{t+1}]^{1/t}k^*$. Similarly, $\xi(k_i) < 0$ when $k_i > [\frac{t-1}{t+1}(M-1)]^{1/t}k^*$, which means that $\Phi(k_i)$ decreases when $k_i > [\frac{t-1}{t+1}(M-1)]^{1/t}k^*$. We next show that there exists a unique $k' \in ([\frac{t-1}{t+1}]^{1/t}k^*, [\frac{t-1}{t+1}(M-1)]^{1/t}k^*)$ such that $\Phi(k_i)$ increases (decreases) if and only if $k_i < (>)\ k'$. For this purpose, it suffices to show that there exists a unique $k' \in ([\frac{t-1}{t+1}]^{1/t}k^*, [\frac{t-1}{t+1}(M-1)]^{1/t}k^*)$, such that $\xi(k') = 0$.

First, such k' must exist by continuity of $\xi(k_i)$. As have been revealed, $\xi(k_i) > 0$ when $k_i < [\frac{t-1}{t+1}]^{1/t}k^*$; and $\xi(k_i) < 0$ when $k_i > [\frac{t-1}{t+1}(M-1)]^{1/t}k^*$.

Second, the uniqueness of k' can be verified as below. We have

$$\begin{aligned}
&\left.\frac{\partial^3 p_i(k_i,\mathbf{k}_{-i};N)}{\partial k_i^3}\right|_{k_{-i}=k^*}\\
&= t(N-1)k^{*t}\left\{\begin{array}{l}(t-2)k_i^{t-3}\frac{-(t+1)k_i^t+(t-1)(N-1)k^{*t}}{[k_i^t+(N-1)k^{*t}]^3}\\ +k_i^{t-2}\frac{-t(t+1)k_i^{t-1}[k_i^t+(N-1)k^{*t}]-3tk_i^{t-1}[-(t+1)k_i^t+(t-1)(N-1)k^{*t}]}{[k_i^t+(N-1)k^{*t}]^4}\end{array}\right\}\\
&= \frac{t(N-1)k^{*t}k_i^{t-3}}{[k_i^t+(N-1)k^{*t}]^3}\left\{\begin{array}{l}(t-2)[-(t+1)k_i^t+(t-1)(N-1)k^{*t}]\\ +\frac{-t(t+1)k_i^t[k_i^t+(N-1)k^{*t}]-3tk_i^t[-(t+1)k_i^t+(t-1)(N-1)k^{*t}]}{[k_i^t+(N-1)k^{*t}]}\end{array}\right\}\\
&= \frac{t(N-1)k^{*t}k_i^{t-3}}{[k_i^t+(N-1)k^{*t}]^3}\left\{\begin{array}{l}(t-2)[-(t+1)k_i^t+(t-1)(N-1)k^{*t}]\\ +\frac{2tk_i^t}{[k_i^t+(N-1)k^{*t}]}[(t+1)k_i^t-(2t-1)(N-1)k^{*t}]\end{array}\right\}.
\end{aligned}$$

Recall $\xi_N(k_i) = \frac{-(t+1)k_i^t+(t-1)(N-1)k^{*t}}{[k_i^t+(N-1)k^{*t}]^3} t k_i^{t-2}(N-1)k^{*t}$. We then have

$$\left.\frac{\partial^3 p_i(k_i, \mathbf{k}_{-i}; N)}{\partial k_i^3}\right|_{k_{-i}=k^*} = (t-2)k_i^{-1}\xi_N(k_i) + \frac{2t^2(N-1)k^{*t}k_i^{2t-3}}{[k_i^t+(N-1)k^{*t}]^4}[(t+1)k_i^t - (2t-1)(N-1)k^{*t}].$$

We now claim $[(t+1)k_i^t - (2t-1)(N-1)k^{*t}]$ is negative for all $k_i \le [\frac{t-1}{t+1}(M-1)]^{1/t}k^*$. A detailed proof is as follows. From $k_i \le [\frac{t-1}{t+1}(M-1)]^{1/t}k^*$, we have $(t+1)k_i^t \le (t-1)(M-1)k^{*t}$. To show $(t+1)k_i^t - (2t-1)(N-1)k^{*t} < 0$, it suffices to show $(t-1)(M-1) < (2t-1)(N-1)$ when $N=2$, which requires $(t-1)(M-1) < (2t-1)$. This always holds when $M = 2, 3$. And $t < 1 + \frac{1}{M-3}$ holds as $t \le 1 + \frac{1}{M-2}$ when $M > 3$.

We thus have at any $k_i \in ([\frac{t-1}{t+1}]^{1/t}k^*, [\frac{t-1}{t+1}(M-1)]^{1/t}k^*)$ such that $\xi(k_i) = 0$, $\xi(k_i)$ must be locally decreasing, because $\frac{\partial \xi(k_i)}{\partial k_i} = (t-2)k_i^{-1}\sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}\xi_N(k_i) + \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}A_N(k_i) = (t-2)k_i^{-1}\xi(k_i) + \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}A_N(k_i) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}A_N(k_i) < 0$ as $A_N(k_i) = \frac{2t^2(N-1)k^{*t}k_i^{2t-3}}{[k_i^t+(N-1)k^{*t}]^4}[(t+1)k_i^t - (2t-1)(N-1)k^{*t}] < 0$.

We are ready to show the uniqueness of k' by contradiction. Suppose that there exists more than one zero points k' and k'' with $k' \ne k''$ for $\xi(k_i)$. Because $\xi(k_i)$ must be locally decreasing, then there must exist at least another zero point $k''' \in (k', k'')$ at which $\xi(k_i)$ is locally increasing. Contradiction thus results. Hence, such a zero point k' of $\xi(k_i)$ must be unique.

Recall $\Phi(k_i)$ increases (decreases) if and only if $k_i < (>)\ k'$ and it reaches its maximum at k'. Note $\frac{\partial \tilde{\pi}_i(k_i)}{\partial k_i} = V\Phi(k_i) - 1$ and $\Phi(0) = 0$. Therefore $\frac{\partial \tilde{\pi}_i(k_i)}{\partial k_i}|_{k_i=0} < 0$. Thus $\frac{\partial \tilde{\pi}_i(k_i)}{\partial k_i}$ has exactly two zero points with the smaller one (k_s) being the local minimum point of $\tilde{\pi}_i(k_i)$. Note $k_i = k^*$ must be a zero point for $\frac{\partial \tilde{\pi}_i(k_i)}{\partial k_i}$ by definition. Since $k_i = k^*$ is a local maximum point of $\tilde{\pi}_i(k_i)$, it is higher than other zero point (k_s) of $\frac{\partial \tilde{\pi}_i(k_i)}{\partial k_i}$ which is a local minimum point of $\tilde{\pi}_i(k_i)$.

Note $x_m = (k_s)^{1/\alpha}$ is the unique local minimum of $\tilde{\pi}_i(x_i)$, and note $x^* = (k^*)^{1/\alpha}$ is the unique inner local maximum of $\tilde{\pi}_i(x_i)$. Note $x_m < x^*$. The results of Lemma 4 are shown. □

Proof of Lemma 5 Proof of Lemma 3 has shown that $F(q, r) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1} (1-q)^{M-N}\frac{V}{N}(1 - \frac{N-1}{N}\frac{r}{\alpha}) - \Delta$ decreases with both q and r. Thus $F(q, r) = 0$ uniquely defines r as a decreasing function of q. Since $F(q_0, r_0) = 0$ and $\hat{q} > q_0$, we must have $r(\hat{q}) < r_0$. □

Proof of Theorem 6 According to Lemma 5, Theorem 4 thus means that contest $r(\hat{q})$ would induce entry equilibrium $\hat{q}$ and pure-strategy bidding whenever $r(\hat{q}) \le \bar{r}$. Since we have a pure-strategy bidding, an overall effort of $\bar{x}_T(\hat{q})$ clearly is induced at the equilibrium.

Consider any other $r \neq r(\hat{q})$. If r induces equilibrium entry $q(r)$ and pure-strategy bidding, then the total effort induced is $\bar{x}_T(q(r))$. Note that by Lemma 3, equilibrium $q(r)$ decreases with r. Thus $r \neq r(\hat{q})$ means $q(r) \neq \hat{q}$. $\bar{x}_T(q)$ is single peaked at $\hat{q}$ according to Theorem 2. Thus for any $r \neq r(\hat{q})$, we must have $\bar{x}_T(q(r)) < \bar{x}_T(\hat{q})$. If r induces equilibrium entry $q(r)$ and mixed-strategy bidding, then the total expected effort induced is strictly lower than $\bar{x}_T(q(r))$ when $\alpha > 1$, based on the arguments deriving this boundary in Sect. 3.4. Therefore the total effort induced must be strictly lower than $\bar{x}_T(\hat{q})$. □

Proof of Theorem 7 We first present two lemmas that lead directly to the main theorem. The proof of Lemma 7 is standard and thus omitted. The proof of Lemma 8 is presented separately.

Lemma 7 *Consider a winner-take-all Tullock contest with impact function x^r, $r \in (0, \alpha)$ and contingent prizes $V_N = V + N \times F$, where V_N is a winner prize that is contingent on the actual number of entrants and $F(> -\frac{V}{M})$ is a uniform entry fee. This contest induces a unique symmetric equilibrium. The equilibrium entry probability q^* satisfies*

$$\sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N} \frac{V_N}{N}\left(1 - \frac{N-1}{N}\frac{r}{\alpha}\right) = \Delta + F. \tag{11}$$

Each participating bidder, upon his entry, places a bid

$$x^* = \left[\sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N} \frac{N-1}{N^2}\frac{rV_N}{\alpha}\right]^{\frac{1}{\alpha}}. \tag{12}$$

With $\frac{r}{\alpha} \leq 1$, the expected payoff function of an entrant is concave, which leads to a unique pure-strategy bidding equilibrium. We further obtain the following.

Lemma 8 *The equilibrium entry probability q^* strictly decreases with entry fee F, with* $\lim_{F\to+\infty} q^* = 0$.

We then prove the theorem. Recall Theorem 2(a), if $\frac{V}{M} \geq \frac{\alpha}{\alpha-1}\Delta$, then $\hat{q} = 1$ can be implementable by some entry fee/subsidy while making sure the payoff of entrants is zero. If $\frac{V}{M} < \frac{\alpha}{\alpha-1}\Delta$, then $\hat{q} \in (q_0, 1)$; We conclude that $\bar{x}_T^*$ can always be achieved through a properly structured Tullock contest. Let $\widetilde{q}$ be the equilibrium entry probability if there are $N > 1$ entrants and each gets an entry subsidy contingent on number of entrants, i.e., $F = -\frac{V}{N}$, clearly $r\ (< r(\hat{q}))$ induces a higher $\widetilde{q}\ (> \hat{q})$ based on similar arguments in the proof of Lemma 8 and equilibrium effort is zero. For this $r \in (0, \alpha)$, Lemma 8 means that there exists a unique entry fee $F\ (> -\frac{V}{M} \geq -\frac{V}{N})$ such that the lemma 7 contest induces equilibrium entry $\hat{q}$. Note that for the Lemma 7 contest with impact function x^r, $r \in (0, \alpha)$ and entry fee F, equation (2) still holds. Since the contest induces entry $\hat{q}$ and pure-strategy bidding, it must induces an expected overall bid of $\bar{x}_T^*$. □

Proof of Lemma 8 Entry equilibrium q^* from (11) is the solution of

$$\Phi(q) = \sum_{N=0}^{M-1} C_{M-1}^{N} q^{N} (1-q)^{(M-1)-N} \zeta_{N+1} = \Delta, \tag{13}$$

where $\zeta_N = \frac{V}{N}(1 - \frac{N-1}{N}\frac{r}{\alpha}) - \frac{N-1}{N}\frac{r}{\alpha}F, \forall N \geq 1$. Let $t = \frac{1}{N}$, then $\zeta_t = tV(1 - (1-t)\frac{r}{\alpha}) - (1-t)\frac{r}{\alpha}F$.

Note $\frac{\partial \zeta_t}{\partial t} = \left(1 - \frac{r}{\alpha}\right) V + (F + 2Vt)\frac{r}{\alpha} > 0$ since $r < \alpha$ and $F > -\frac{V}{M} > -\frac{2V}{N}$. Thus ζ_N strictly decreases with N.

$$\Phi'(q) = (M-1) \sum_{N=0}^{M-2} C_{M-2}^{N} q^{N} (1-q)^{(M-2)-N} (\zeta_{N+2} - \zeta_{N+1}) < 0.$$

Since $\Phi'(q) < 0, \forall q > 0$ and $\Phi'(F) < 0, \forall F \geq -\frac{V}{M}$, we must have $\frac{dq}{dF} < 0$. When F is big, only ζ_1 is positive and ζ_N, $N \geq 2$ are very negative. Thus $\Phi(q) = \Delta$ means a very small q. □

Proof of Lemma 6 By definition, $\bar{x}_T^*\left(M'\right) = \bar{x}_T(\hat{q}(M'); M')$

$$= \begin{cases} (M'\hat{q}(M'))^{\frac{\alpha-1}{\alpha}} \left\{[1-(1-\hat{q}(M'))^{M'}]V - M'\hat{q}(M')\Delta\right\}^{\frac{1}{\alpha}} & \text{when } \frac{V}{M} < \frac{\alpha}{\alpha-1}\Delta \\ M'\left(\frac{V}{M'} - \Delta\right)^{\frac{1}{\alpha}} & \text{when } \frac{V}{M} \geq \frac{\alpha}{\alpha-1}\Delta. \end{cases}$$

When $\frac{V}{M} < \frac{\alpha}{\alpha-1}\Delta$, $\hat{q}(M') \in (0,1)$. By Envelope Theorem, $\frac{d\bar{x}_T(\hat{q}(M');M')}{dM'} = \frac{\partial \bar{x}_T(q;M')}{\partial M'}|_{q=\hat{q}(M')}$. We have

$$\begin{aligned} &\frac{\partial \bar{x}_T(q;M')}{\partial M'}|_{q=\hat{q}(M')} \\ &= \partial\left[(M'\hat{q}(M'))^{\frac{\alpha-1}{\alpha}} \left\{[1-(1-\hat{q}(M'))^{M'}]V - M'\hat{q}(M')\Delta\right\}^{\frac{1}{\alpha}}\right]/\partial M' \\ &= \frac{\alpha-1}{\alpha} M'^{-\frac{1}{\alpha}} \left[\hat{q}(M')\right]^{\frac{\alpha-1}{\alpha}} \left\{[1-(1-\hat{q}(M'))^{M'}]V - M'\hat{q}(M')\Delta\right\}^{\frac{1}{\alpha}} \\ &\quad + \frac{1}{\alpha}(M'\hat{q}(M'))^{\frac{\alpha-1}{\alpha}} \left\{[1-(1-\hat{q}(M'))^{M'}]V - M'\hat{q}(M')\Delta\right\}^{\frac{1}{\alpha}-1} \\ &\quad \times[-(1-\hat{q}(M'))^{M'} V \ln(1-\hat{q}(M')) - \hat{q}(M')\Delta], \end{aligned}$$

which has the same sign as

$$\begin{aligned} \lambda = (\alpha-1)&\left\{[1-(1-\hat{q}(M'))^{M'}]V - M'\hat{q}(M')\Delta\right\} \\ &+ M'[-(1-\hat{q}(M'))^{M'} V \ln(1-\hat{q}(M')) - \hat{q}(M')\Delta]. \end{aligned}$$

Because $-\ln(1-\hat{q}(M^{'})) < \frac{\hat{q}(M^{'})}{1-\hat{q}(M^{'})}$, we have $M^{'}[-(1-\hat{q}(M^{'}))^{M^{'}}V\ln(1-\hat{q}(M^{'})) - \hat{q}(M^{'})\Delta] < \hat{q}(M^{'})[M^{'}(1-\hat{q}(M^{'}))^{M^{'}-1}V - M^{'}\Delta]$. Hence, $\lambda < (\alpha - 1)\left\{[1-(1-\hat{q}(M^{'}))^{M^{'}}]V - M^{'}\hat{q}(M^{'})\Delta\right\} + \hat{q}(M^{'})[M^{'}(1-\hat{q}(M^{'}))^{M^{'}-1}V - M^{'}\Delta] = 0$ (by the definition of $\hat{q}(M^{'})$). We then have $\frac{d\bar{x}_T(\hat{q}(M^{'});M^{'})}{dM^{'}} < 0$.

When $\frac{V}{M} \geq \frac{\alpha}{\alpha-1}\Delta$, $\hat{q}(M^{'}) = 1$. We have

$$\frac{\partial \bar{x}_T^*\left(M^{'}\right)}{\partial M^{'}} = \left(\frac{V}{M^{'}} - \Delta\right)^{\frac{1}{\alpha}} - \frac{V}{\alpha}\frac{1}{M^{'}}\left(\frac{V}{M^{'}} - \Delta\right)^{\frac{1}{\alpha}-1},$$

$$\frac{\partial^2 x_T^*\left(M^{'}\right)}{\partial M^{'2}} = \frac{V^2}{\alpha}\left(\frac{1}{\alpha}-1\right)\frac{1}{M^{'3}}\left(\frac{V}{M^{'}} - \Delta\right)^{\frac{1}{\alpha}-2} < 0.$$

Since $\overline{x}_T^*\left(M^{'}\right)$ is concave in $M^{'}$. $\overline{x}_T^*\left(M^{'}\right)$ achieves its maximum value

$$\overline{x}_T^*\left(\widehat{M^{'}}\right) = \left(1-\frac{1}{\alpha}\right)\frac{V}{\Delta}\left(\frac{V}{\left(1-\frac{1}{\alpha}\right)\frac{V}{\Delta}} - \Delta\right)^{\frac{1}{\alpha}}$$

when $M^{'} = \widehat{M^{'}} = \left(1-\frac{1}{\alpha}\right)\frac{V}{\Delta}$. Therefore, $\bar{x}_T^*\left(M^{'}\right)$ increases with $M^{'}$ when $M^{'} \leq \left(1-\frac{1}{\alpha}\right)\frac{V}{\Delta}$. □

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