

# Effective Zero-Inventory-Ordering Policies for the Single-Warehouse Multiretailer Problem with Piecewise Linear Cost Structures

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We analyze the problem faced by companies that rely on TL (Truckload) and LTL (Less than Truckload) carriers for the distribution of products across their supply chain. Our goal is to design simple inventory policies and transportation strategies to satisfy time-varying demands over a finite horizon, while minimizing systemwide cost by taking advantage of quantity discounts in the transportation cost structures. For this purpose, we study the cost effectiveness of restricting the inventory policies to the class of zero-inventory-ordering (ZIO) policies in a single-warehouse multiretailer scenario in which the warehouse serves as a cross-dock facility. In particular, we demonstrate that there exists a ZIO inventory policy whose total inventory and transportation cost is no more than  $4/3$  ( $5.6/4.6$  if transportation costs are stationary) times the optimal cost. However, finding the best ZIO policy is an NP-hard problem as well. Thus, we propose two algorithms to find an effective ZIO policy: An exact algorithm whose running time is polynomial for any fixed number of retailers, and a linear-programming-based heuristic whose effectiveness is demonstrated in a series of computational experiments. Finally, we extend the worst-case results developed in this paper to systems in which the warehouse does hold inventory.

*(Approximation Algorithms; Zero-Inventory-Ordering Policies; Inventory; Transportation; Truckload (TL); Less than truckload (LTL))*

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## 1. Introduction

In recent years, many companies have realized that important cost savings can be achieved by integrating inventory control and transportation policies throughout their supply chains. Thus, the problem faced by these companies is to find an optimal replenishment

plan, i.e., an inventory and transportation strategy, so as to minimize total inventory and transportation costs over a finite planning horizon. The difficulty in designing a coordinated strategy, however, is compounded by the fact that typically these companies rely on external third-party logistics providers for the

transportation of goods from suppliers through warehouses to retailers.

These problems are different from traditional network flow problems, as the transportation cost structure, also referred to as *ordering cost*, offered by the carriers is usually piecewise linear but not necessarily convex. This cost structure, representing quantity discounts, volume-based price incentives, and other forms of economies of scale, has a major impact on the replenishment strategy. It usually reflects either incremental or all-unit discount effects, leading to the following types of cost functions.

**Incremental Discount Cost Structures.** This class can be fully characterized by piecewise linear concave cost functions. Of course, a special case of this cost function is the fixed-charge function, where a fixed cost, independent of the shipment size, is incurred whenever there is a shipment.

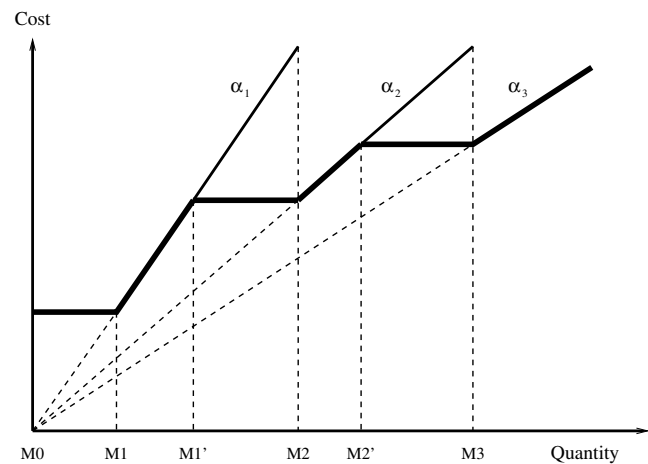
**All-Unit Discount Cost Structures.** Such a cost function implies that if the facility orders  $Q$  units, the transportation cost function is

$$G(Q) = \begin{cases} 0 & \text{if } Q=0, \\ c & \text{if } 0 < Q < M_1, \\ \alpha_1 Q & \text{if } M_1 \leq Q < M_2, \\ \alpha_i Q & \text{if } M_i \leq Q < M_{i+1}, \quad i=2,3,\dots,n-1, \\ \alpha_n Q & \text{if } M_n \leq Q, \end{cases}$$

where  $\alpha_1 > \alpha_2 > \dots \geq 0$ , and  $\alpha_1 M_1 = c$ . Thus,  $c$  is a *minimum charge* for shipping a small volume, i.e.,  $c$  is the total cost when the number of units shipped is no more than  $M_1$ . Interestingly, in practice, when the shipper is planning to ship  $Q$  units,  $M_i \leq Q < M_{i+1}$ , the cost is calculated as  $F(Q) = \min\{G(Q), G(M_{i+1})\} = \min\{\alpha_i Q, \alpha_{i+1} M_{i+1}\}$ . That is, if the order quantity is greater than a certain value, the shippers pay as if they were shipping  $M_{i+1}$  units. This is called in the industry *shipping  $Q$  but declaring  $M_{i+1}$* .

This commonly used practice implies that the true transportation cost function,  $F(\cdot)$ , has the structure described by the heavy solid line in Figure 1. As the dashed lines indicate, the associated solid lines originate at point  $(0, 0)$ . We refer to such cost functions as *modified all-unit discount cost functions*. Notice that these cost functions satisfy the following properties:

**Figure 1** Modified All-Unit Discount Cost Structure



(i) they are nondecreasing functions of the amount shipped, and (ii) the cost per unit is nonincreasing in the amount shipped. Indeed, these two properties are necessary and sufficient to derive our results.

These cost functions are common in industry practice. Most LTL carriers use an industry standard transportation-rating engine called CZAR (Southern Motor Carrier's Complete Zip Auditing and Rating engine). This engine allows the shipper to find the transportation cost of every shipment which is a function of the source, destination, product class, and discount. The carrier and the shipper contractually agree on the product class (typically *class 100*) and on the level of discount, which implies that the shipper will pay only a given fraction, say 90%, of the cost generated by the rating engine. Now, given this input, the transportation cost as a function of the amount shipped enjoys the all-unit discount structure described above.

In this paper, we study a class of multiperiod distribution problems with transportation cost structures that model both the incremental and all-unit discount cost functions. Specifically, we consider a classical inventory distribution model in which a single warehouse receives inventory from a single supplier and replenishes the inventory of  $n$  retailers. Each retailer provides the warehouse with forecast demand for the next  $T$  time periods.

We assume that shortages and backlogging are not allowed either at the warehouse or at the retailers.

Furthermore, we assume that the warehouse uses a common logistic strategy, referred to as cross-docking, in which the warehouse acts merely as a coordinator of the supply process and as a transshipment point for incoming orders from the supplier, but does not hold any stocks.

In these situations, (large) shipments from the supplier to the warehouse are often delivered by TL carriers whose costs can be approximated by piecewise linear concave functions (for example, a fixed-charge cost function). Economies of scale in production can also be modeled in this way. Henceforth, we will assume that the ordering cost function from the supplier to the warehouse is of the incremental discount type. By contrast, since shipment sizes from the warehouse to a retailer are relatively small, these shipments are typically delivered by LTL carriers whose costs follow the modified all-unit discount cost structure. The objective is to find an optimal shipment plan that exploits the quantity discount effect and, at the same time, controls the inventory holding cost at the retailers' end.

Observe that the single-warehouse multiretailer problem described here can also be used to model the *joint replenishment problem*; see Joneja (1990). In this problem, a single facility replenishes a set of items over a finite horizon. Whenever the facility places an order for a subset of the items, two types of costs are incurred: a joint set-up cost and an item-dependent set-up cost. The objective in the joint replenishment problem is to decide when and how many units to order for each item so as to minimize inventory-holding and ordering costs over the planning horizon. Because the joint replenishment problem is NP-hard (see Arkin et al. 1989), the single-warehouse multiretailer problem is also NP-hard even if all transportation cost functions are fixed-charge cost functions.

Evidently, the fixed-charge cost function is a special case of the all-unit discount cost function considered in this paper. This implies that the problem analyzed in this paper is NP-hard in general. An interesting question is whether it is NP-hard for a single or fixed number of retailers. This question was answered by Chan et al. (2002), who show that a special case of our problem, in which a single retailer is replenished by a single warehouse with zero transportation

cost for shipments to the warehouse and modified all-unit discount transportation costs for shipments to the retailer, is NP-hard. Thus, the single-warehouse multiretailer problem described above is NP-hard *even for a fixed number of retailers*.

Our focus in this paper is on a class of policies referred to as zero-inventory-ordering (ZIO) policies, in which orders are placed by the retailers only when their inventory levels drop to zero. It is easy to see that in the case of concave transportation cost functions there exists an optimal ZIO policy for the single-warehouse multiretailer problem. Unfortunately, the result of Arkin et al. (1989) implies that finding the best ZIO is in itself NP-hard. Of course, our model is more general and, as shown by Chan et al. (2002) for the single-warehouse single-retailer case, the optimal policy may not be a ZIO policy.

This paper builds upon results developed in Chan et al. (1999), which considers more general production-distribution problems with piecewise linear concave costs, and in Chan et al. (2002), which studies a simpler dynamic lot-sizing problem with a single retailer and modified all-unit discounts. In §3 we extend the results in Chan et al. (2002) to the one-warehouse multiretailer case and show that (i) there is a ZIO policy whose associated cost is no more than  $4/3$  times the cost of the optimum replenishment plan, and (ii) if the ordering cost function does not vary over time, then the cost of the optimal ZIO policy is no more than  $5.6/4.6$  times the optimal cost. This leads to the development of efficient algorithms. Section 4 describes an exact algorithm that finds the best ZIO policy and whose running time is polynomial for a fixed number of retailers. The second algorithm, described in §5, is based on formulating the problem of finding the best ZIO policy as an integer program. This integer program has a similar structure to the one developed in Chan et al. (1999) for the concave case. In fact, we find that the properties of the linear programming relaxation derived in that paper can be applied to our new integer programming formulation as well. The linear programming relaxation of this model is consequently solved and its solution and structural properties are used to generate a ZIO policy. An empirical study shows (see §6) that the heuristic algorithm is computationally efficient and generates solutions very close to the optimal ZIO policy.

Finally, in §7 we point out extensions of the worst-case results to more general cost structures and to the case of a traditional inventory distribution system in which the warehouse may hold inventory.

## 2. Notation and Main Results

Let  $n$  be the number of retailers served by the warehouse and  $T$  be the length of the planning horizon under consideration. For each  $t = 1, 2, \dots, T$ , we let  $K_t^0(Q_t^0)$  be the piecewise linear concave transportation cost function associated with shipping a quantity  $Q_t^0$  from the supplier to the warehouse at time  $t$ . Similarly, for each  $i = 1, 2, \dots, n$  and  $t = 1, 2, \dots, T$ , we denote by  $K_t^i(Q_t^i)$  the modified all-unit discount transportation cost function associated with shipping a quantity  $Q_t^i$  from the warehouse to retailer  $i$  at time  $t$ . Finally, for each  $i = 1, 2, \dots, n$  and  $t = 1, 2, \dots, T$ , let  $I_t^i$  denote the inventory at retailer  $i$  at the end of period  $t$ ,  $h_t^i$  the cost of holding an item at retailer  $i$  at the end of period  $t$ , and  $d_t^i$  the demand of retailer  $i$  at time  $t$ .

Our objective is to find the size and timing of shipments so as to minimize total transportation and inventory costs while satisfying all demands without shortages. In what follows, we will refer to this problem as the *single-warehouse multiretailer problem* (SWMR):

$$\begin{aligned} \text{Problem SWMR: } \quad & \text{Min} \quad \sum_{t=1}^T \left[ K_t^0(Q_t^0) \right. \\ & \quad \left. + \sum_{i=1}^n (K_t^i(Q_t^i) + h_t^i I_t^i) \right] \\ \text{s.t.} \quad & Q_t^0 = \sum_{i=1}^n Q_t^i \quad \forall t = 1, 2, \dots, T, \\ & Q_t^i + I_{t-1}^i = d_t^i + I_t^i \quad \forall i = 1, \dots, n, \quad t = 1, 2, \dots, T, \\ & I_0^i = 0 \quad \forall i = 1, 2, \dots, n, \\ & Q_t^i \geq 0 \quad \forall i = 0, 1, \dots, n, \quad t = 1, 2, \dots, T, \\ & I_t^i \geq 0 \quad \forall i = 1, \dots, n, \quad t = 1, 2, \dots, T, \quad (1) \end{aligned}$$

where without loss of generality we assume that initial inventory is 0; that is,  $I_0^i = 0$ , for all  $i = 1, 2, \dots, n$ .

Let  $Z^*$  be the cost of the optimal solution to the single-warehouse multiretailer problem and for any heuristic  $H$ , let  $Z^H$  be the cost of the solution generated by heuristic  $H$ .

We first show that unless  $P = NP$ , it is not possible to develop an algorithm that runs in polynomial time and generates, for any instance of the problem, a solution which is within a factor of  $O(\log n)$  from optimality.

**THEOREM 2.1.** *Suppose there exists a  $\gamma > 0$  and a polynomial time heuristic,  $H$ , for the single-warehouse multiretailer problem such that for all instances*

$$\frac{Z^H}{Z^*} \leq \gamma \log n,$$

then  $P = NP$ .

**PROOF.** The proof is based on showing that the set-covering problem can be reduced to the single-warehouse multiretailer problem. It is well known (see Feige 1998 or Arora and Sudan 1997) that there is no polynomial time algorithm for the set-covering problem with worst-case bound better than  $\gamma \log n$ , for  $\gamma > 0$ , unless  $P = NP$ . Consider an instance of the set-covering problem:  $\min(\sum_{i=1}^m x_i : Ax \geq 1)$ , where  $A = (a_{i,t})$  is a  $n \times m$  0-1 matrix. It can be reduced to the single-warehouse multiretailer problem with  $n$  retailers and  $m+1$  periods as follows. Let

$$K_t^i(x) = \begin{cases} M\delta(x) & \text{if } a_{i,t} = 0 \\ 0 & \text{if } a_{i,t} = 1 \end{cases} \quad \text{for all } i \text{ and } t = 1, 2, \dots, m,$$

$$K_{m+1}^i(x) = M\delta(x) \quad \text{for all } i,$$

$$K_t^0(x) = \delta(x) \quad \text{for } t = 1, 2, \dots, m, m+1,$$

$$d_t^i = \begin{cases} 0 & \text{if } t = 1, 2, \dots, m \\ 1 & \text{if } t = m+1 \end{cases} \quad \text{for all } i,$$

$$h_t^i = 0 \quad \text{for all } i, t,$$

where  $M$  is some large number no less than  $m$ , and  $\delta(x) = 1$  when  $x > 0$ , and 0 otherwise.

The high set-up cost at time  $m+1$  forces retailers to order in earlier periods. In addition, an order for retailer  $i$  is placed at time  $t$  only if  $a_{i,t} = 1$  because there is a large fixed cost associated with shipments

in periods in which  $a_{i,t} = 0$ . Thus, finding the best inventory ordering policy in this situation is equivalent to finding the minimum number of ordering periods, which is determined by clustering retailers that will be served together at a certain time.  $\square$

Thus, in the remainder of this paper we focus on the analysis of a class of policies referred to as zero-inventory-ordering (ZIO) policies. In this class, orders are placed only at times when on-hand inventory has been fully depleted; that is,  $Q_i I_{i-1}^i = 0$ . Let  $Z^{\text{ZIO}}$  be the cost associated with the optimal ZIO policy. Of course, the optimal policy for the single-warehouse multiretailer problem may not be a ZIO policy. However, in §3 we show the following two theorems.

**THEOREM 2.2.** *For every instance of the single-warehouse multiretailer problem,*

$$Z^{\text{ZIO}} \leq \frac{4}{3} Z^*,$$

and this bound is tight.

In practice, the ordering cost function does not vary from period to period; i.e., for all  $t$ ,  $K_i^0(\cdot) = K^0(\cdot)$  and  $K_i^i(\cdot) = K^i(\cdot)$ ,  $i = 1, 2, \dots, n$ . In this case, we show that the worst-case ratio of the cost of the best ZIO policy to the optimal cost is no more than  $5.6/4.6 \approx 1.22$ . That is,

**THEOREM 2.3.** *For every instance of the single-warehouse multiretailer problem in which the transportation cost functions are stationary,*

$$Z^{\text{ZIO}} \leq \frac{5.6}{4.6} Z^*.$$

The optimal ZIO policy can be found in polynomial time for any fixed number of retailers, using the algorithm presented in §4. Not surprisingly, however, the computational complexity of this method grows exponentially as the number of retailers increases. To overcome this problem, in §5 we propose a linear-programming-based heuristic that runs in polynomial time. This algorithm is shown to be very efficient in our computational study.

### 3. The Effectiveness of Zero-Inventory-Ordering Policies

In this section we analyze the effectiveness of ZIO policies and prove Theorems 2.2 and 2.3. We start by

showing some structural properties of feasible solutions to the single-warehouse multiretailer problem.

Let  $S$  be a feasible replenishment plan for the system. Let  $m$  be the number of shipments from the supplier to the warehouse and  $T^0 = \{t_1 = 1, t_2, t_3, \dots, t_m\}$  be the time epochs in which these shipments occur. Similarly, let  $m_i$  be the number of orders placed by retailer  $i$ ,  $i = 1, 2, \dots, n$ , and  $T^i = \{t_1^i = 1, t_2^i, \dots, t_{m_i}^i\} \subset T^0$  be the time epochs in which goods are delivered from the warehouse to retailer  $i$ . Let  $Q_t^i$  denote the size of the shipment to the retailer at time period  $t$  and  $Q_t^0 = \sum_{i=1}^n Q_t^i$  be the quantity required at the warehouse. Obviously,  $Q_t^i = 0$  for all  $t \notin T^i$ ,  $i = 0, 1, 2, \dots, n$ . Observe that all these parameters are associated with the inventory policy  $S$  under consideration. We omit the reference to  $S$  in our notation because it is clear what policy they are referring to at any point. However, we add the superscript  $*$  to the parameters corresponding to an optimal policy,  $S^*$ .

Consider the inventory level at retailer  $i$ . Without loss of generality, we can assume that in any replenishment plan, the shipment that arrived at time  $t_{l+1}^i$  will only be used *after* the depletion of the shipment that arrived at time  $t_l^i$ . That is, without loss of generality we assume that orders are used to satisfy demand in a first-in-first-out basis. Thus, let  $s_l^i$ ,  $i = 1, 2, \dots, n$  and  $l = 1, 2, \dots, m_i$ ,  $s_l^i \geq t_l^i$ , be the earliest (or first) time a shipment that arrives in time  $t_l^i$  is used to satisfy customer demand. Observe that in the optimal strategy, a shipment that arrives at time  $t_l^i$  may only be needed at time  $s_l^i$  ( $s_l^i > t_l^i$ ). The early shipment may be due to the need to exploit the characteristics of the transportation cost function to the warehouse or the retailer. That is, the early delivery may take advantage of changes in the cost function as a function of time, and/or the effect of the transportation discount on the order quantity.

Consider retailer  $i$  and let  $H_k^i$  denote the cost of holding an item at retailer  $i$  from time  $t_k^i$  to the beginning of period  $t_{k+1}^i$ . That is,

$$H_k^i \equiv h_{t_k^i}^i + h_{(t_k^i+1)}^i + \dots + h_{(t_{k+1}^i-1)}^i.$$

In addition, let

$$A_k^i \equiv \frac{K_{t_k^i}^0(Q_{t_k^i}^0) - K_{t_k^i}^0(Q_{t_k^i}^0 - Q_{t_k^i}^i) + K_{t_k^i}^i(Q_{t_k^i}^i)}{Q_{t_k^i}^i}.$$

**OBSERVATION 3.1.**  $A_k^i$  represents the ordering cost per unit associated with shipping the quantity  $Q_{t_k^i}^i$  to retailer  $i$  at period  $t_k^i$ , and it is also an upper bound on the unit cost associated with ordering additional units in that period for retailer  $i$ .

This is due to the concavity of  $K_{t_k^i}^0(\cdot)$  and the fact that cost per unit resulting from the LTL charges is nonincreasing with volume.

**LEMMA 3.2.** Given any feasible policy  $S$ , there exists a feasible policy with lower or equal cost in which a positive fraction of any order is used to satisfy demand for periods prior to the arrival of the subsequent order. That is,  $t_{k+1}^i > s_k^i \geq t_k^i$  for all  $i = 1, 2, \dots, n$  and  $k = 1, 2, \dots, m_i$ .

**PROOF.** Suppose that the current policy  $S$  does not satisfy this property; i.e., there exists an index  $k$  such that  $t_{k+1}^i \leq s_k^i$ . Because both the orders at periods  $t_k^i$  and  $t_{k+1}^i$  cover demand occurring on or after period  $t_{k+1}^i$ , the two orders can be combined and sent in the same period, either  $t_k^i$  or  $t_{k+1}^i$ . The total costs associated with ordering the combined quantity and holding the units in inventory until period  $t_{k+1}^i$  is no more than

$$\begin{aligned} & (Q_{t_k^i}^i + Q_{t_{k+1}^i}^i) \min\{A_k^i + H_k^i, A_{k+1}^i\} \\ & \leq Q_{t_k^i}^i (A_k^i + H_k^i) + Q_{t_{k+1}^i}^i A_{k+1}^i, \end{aligned}$$

which is the cost associated with ordering those quantities and holding the units in inventory until period  $t_{k+1}^i$  in the current policy  $S$ . Since all other costs remain the same when combining the orders, the above argument shows that we can always obtain a policy with lower or equal cost satisfying the property.  $\square$

Given a policy  $S$  satisfying the condition in Lemma 3.2, the order placed by retailer  $i$  in any period  $t_j^i$  can be written as

$$Q_{t_j^i}^i = \alpha_j^i Q_{t_j^i}^i + (1 - \alpha_j^i) Q_{t_{j+1}^i}^i,$$

where  $\alpha_j^i Q_{t_j^i}^i$ ,  $0 < \alpha_j^i \leq 1$ , denotes the portion of the  $j$ th shipment that is used to satisfy demands from some time  $s_j^i < t_{j+1}^i$  until the arrival of the  $(j+1)$ th shipment. This is used in proving Theorems 2.2 and 2.3, which are the subjects of the subsequent sections.

### 3.1. Time-Varying Ordering Cost Functions

In what follows we show that given an optimal policy of the one-warehouse multiretailer problem we can construct a ZIO policy whose cost is no more than  $4/3$  times the cost of the original solution. For this purpose, we first show that we can focus on a subset of feasible policies.

**LEMMA 3.3.** Given any feasible policy  $S$ , there exists a feasible policy with lower or equal cost satisfying the following properties.

1. A fraction of any order is used to satisfy demand for periods prior to the arrival of the subsequent order. That is,  $t_{k+1}^i > s_k^i \geq t_k^i$  for all  $i = 1, 2, \dots, n$  and  $k = 1, 2, \dots, m_i$ .
2. If the order at  $t_k^i$  covers demands occurring after the next order has arrived at period  $t_{k+1}^i$ , then the cost per unit associated with ordering and holding those units in inventory in the earlier period is higher. That is,  $A_k^i + H_k^i > A_{k+1}^i$ .

**PROOF.** The first property is that in Lemma 3.2. If the second property does not hold, then following the same reasoning as in Lemma 3.2, the two orders can be consolidated and delivered in period  $t_k^i$  without increasing total costs.  $\square$

For  $i = 1, 2, \dots, n$  and  $j = 1, 2, \dots, m_i$ , let

$$C_j^i = \frac{K_{t_j^i}^0(\sum_{x=i}^n Q_{t_j^i}^x) - K_{t_j^i}^0(\sum_{x=i+1}^n Q_{t_j^i}^x) + K_{t_j^i}^i(Q_{t_j^i}^i)}{Q_{t_j^i}^i}.$$

In what follows, we allocate an ordering cost of  $C_j^i$  to each unit of the  $j$ th order of retailer  $i$ . Using this notation and adding a superscript  $*$  to denote values at optimality, the total cost associated with an optimal policy  $S^*$ , satisfying the conditions in Lemma 3.3, can be written as:

$$\begin{aligned} Z^* = \sum_{i=1}^n & \left[ C_1^{*i} \alpha_1^{*i} Q_{t_1^i}^{*i} + \sum_{j=2}^{m_i^*} ((C_{j-1}^{*i} + H_{j-1}^{*i})(1 - \alpha_{j-1}^{*i}) Q_{t_{j-1}^i}^{*i} \right. \\ & \left. + C_j^{*i} \alpha_j^{*i} Q_{t_j^i}^{*i}) \right] + H^*, \end{aligned} \quad (2)$$

where  $H^*$  = total holding cost of policy  $S^*$  minus the cost of carrying, for each  $i = 1, 2, \dots, n$  and  $j = 1, \dots, m_i^* - 1$ , the portion of demand delivered in period  $t_j^{*i}$  but used only after  $t_{j+1}^{*i}$ , from period  $t_j^{*i}$  to  $t_{j+1}^{*i}$ . That is,  $H^* = (\text{total holding cost of policy } S^*) - \sum_{i=1}^n \sum_{j=1}^{m_i^*-1} H_j^{*i}$ .

We now transform the optimal policy  $S^*$  into a ZIO policy  $S$  whose cost is within  $4/3$  of the optimal cost (2). For that purpose, we apply the following procedure.

**Transformation Procedure.**

• *Step 0.* Let  $S = S^*$ .

• *Step 1.* Find the smallest index of a retailer, say retailer  $i$ , that does not satisfy the ZIO policy and the smallest index,  $k$ , such that  $\alpha_{k-1}^i < 1$ ; that is,  $t_k^i$  is the earliest period in which retailer  $i$  places an order before inventory has been fully depleted.

• *Step 2.* Either,

*Combine 1:* Move  $(1 - \alpha_{k-1}^i)Q_{t_{k-1}}^i$  from the order at period  $t_{k-1}^i$  to that at period  $t_k^i$ , or

*Combine 2:* Move  $\alpha_k^i Q_{t_k}^i$  from the order at period  $t_k^i$  to that at period  $t_{k-1}^i$  and  $(1 - \alpha_k^i)Q_{t_k}^i$  from the order at period  $t_k^i$  to that at period  $t_{k+1}^i$ , whichever results in a lower cost.

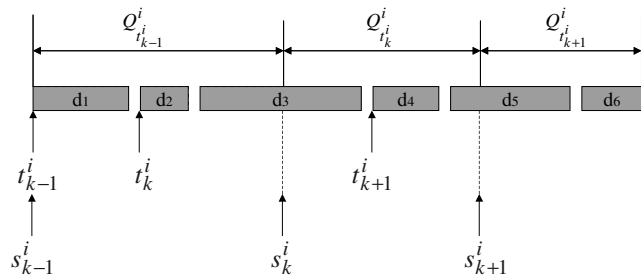
• *Step 3.* If necessary, combine orders without increasing total cost until the current policy satisfies the conditions in Lemma 3.3.

• *Step 4.* Repeat Steps 1 to 3 until all retailers satisfy the ZIO property. Note that this requires at most  $nT$  iterations because the solution generated after each iteration ensures that one more ordering period at a retailer satisfies the ZIO property.

See Figures 2, 3, and 4 for illustration.

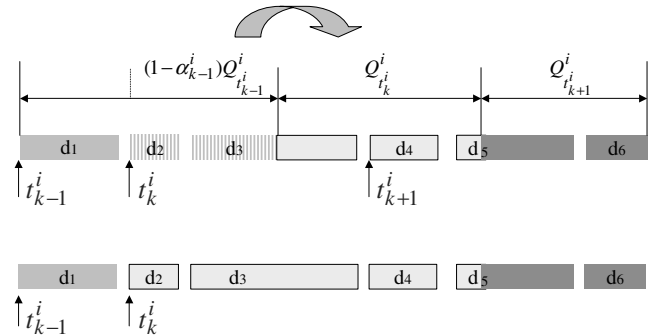
Observe that before the execution of Step 2 for a certain index  $k$  and retailer  $i$ , the total order quan-

**Figure 2** Initial Policy



*Note.* Each shaded block represents the demand (here denoted as  $d_1, d_2, \dots$ ) faced by retailer  $i$  at subsequent time periods, from period  $t_{k-1}^i$  onwards. The size of the block depicts the magnitude of the corresponding demand. The current policy  $S$  is superimposed in the picture by pointing out the periods,  $t_{k-1}^i, t_k^i, t_{k+1}^i$ , at which orders are placed and the particular demands that each of the ordered quantities ( $Q_{t_{k-1}}^i, Q_{t_k}^i, Q_{t_{k+1}}^i$ ) will cover.

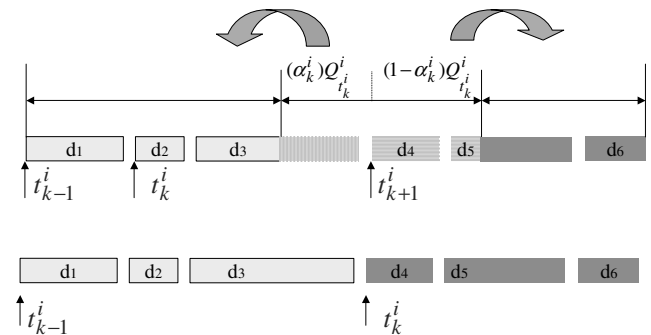
**Figure 3** Policy Obtained if Combine 1 Is Performed



*Note.* The top depicts the combine operation: Add the striped part of the order at period  $t_{k-1}^i$  to that at period  $t_k^i$ . The bottom figure represents the new policy obtained, with demand satisfied by each order in a different shade of gray.

tity at any period  $t \geq t_{k-1}^i$  in the current policy  $S$  satisfies  $Q_t^0 \geq \sum_{x=i}^n Q_t^{*x}$ . This is true because all the orders of retailers  $i + 1$  through  $n$ , and those of  $i$  at periods greater than  $t_{k-1}^i$ , have not been modified and the order of retailer  $i$  at period  $t_{k-1}^i$  may have been increased in the previous iteration. Note that if combine 2 is performed for a previous index  $l$ , the order at period  $t_l^i$  is reduced but  $t_{l+1}^i$  will not be considered in Step 1 of the subsequent stages as the inventory (ordered at  $t_{l-1}^i$ ) would have been fully depleted exactly at period  $t_{l+1}^i$ . Furthermore, even though the order at period  $t_{l+1}^i$  may have been increased, the portion that will be used after period

**Figure 4** Policy Obtained if Combine 2 Is Performed



*Note.* The top depicts the combine operation: Add the vertically striped part of the order at period  $t_k^i$  to that at period  $t_{k-1}^i$  and the horizontally striped part of the order at period  $t_k^i$  to that at period  $t_{k+1}^i$ . The bottom figure represents the new policy obtained, with demand satisfied by each order in a different shade of gray.

$t_{l+2}^i$  (i.e.,  $(1 - \alpha_{l+1}^i)Q_{t_{l+1}^i}^i$ ) remains unchanged under the modified policy.

Note that at each iteration of the transformation procedure no new ordering periods are added. Thus, we can consider without loss of generality that the set of ordering periods  $t_k^i$  remains the same over the iterations; i.e.,  $t_k^i = t_k^{*i}$  for  $k = 1, 2, \dots, m_i^*$  and  $i = 1, 2, \dots, n$ , even if the order quantity in some of those periods becomes 0 after Steps 2 and 3 are executed. The following two lemmas demonstrate that at each iteration of Step 2 for a particular index  $k$  and retailer  $i$ , the increase in cost accrued is no more than one-third of the corresponding  $k$ th term associated with the retailer  $i$  under consideration in the expression of the optimal cost (Equation (2)),

$$\left[ (C_{k-1}^{*i} + H_{k-1}^{*i})(1 - \alpha_{k-1}^{*i})Q_{t_{k-1}^{*i}}^{*i} + C_k^{*i} \alpha_k^{*i} Q_{t_k^{*i}}^{*i} \right].$$

This proves that the cost of the ZIO policy generated by the transformation procedure is no more than  $\frac{4}{3}Z^*$ , since the index  $k$  is strictly increasing in the number of iterations performed and the sum of those terms for all  $k$  and all  $i$  is no larger than the optimal value,  $Z^*$ . Thus, the cost of the optimal ZIO policy satisfies the  $\frac{4}{3}Z^*$  bound, as stated in Theorem 2.2.

LEMMA 3.4. *Let*

$$A_x^i = \frac{K_{t_x^i}^0(Q_{t_x^i}^0) - K_{t_x^i}^0(Q_{t_x^i}^0 - Q_{t_x^i}^i) + K_{t_x^i}^i(Q_{t_x^i}^i)}{Q_{t_x^i}^i}$$

and

$$H_x^i = \sum_{t=t_x^i}^{t_{x+1}^i-1} h_t^i.$$

The increase in cost at the combine step for any retailer  $i$  and index  $k$  is no more than

1.  $A_k^i(1 - \alpha_{k-1}^i)Q_{t_{k-1}^i}^i$  if combine 1 is executed,
2.  $(A_{k-1}^i + H_{k-1}^i - A_k^i)\alpha_k^i Q_{t_k^i}^i$  if combine 2 is executed.

PROOF.

1. The cost increase is not more than  $A_k^i(1 - \alpha_{k-1}^i)Q_{t_{k-1}^i}^i$  if combine 1 is executed. This is true because  $A_k^i$  is an upper bound on the cost per unit associated with extra units sent at period  $t_k$ ; see Observation 3.1.

2. The cost increase is not more than  $(A_{k-1}^i + H_{k-1}^i - A_k^i)\alpha_k^i Q_{t_k^i}^i$  if combine 2 is executed.

We prove this in three steps.

(i) By reducing the quantity ordered at period  $t_k^i$  by  $Q_{t_k^i}^i$ , ordering costs in that period are reduced by exactly  $A_k^i Q_{t_k^i}^i$ , by definition of  $A_k^i$ .

(ii) Moving  $\alpha_k^i Q_{t_k^i}^i$  from period  $t_k^i$  to  $t_{k-1}^i$  increases ordering costs and inventory costs at period  $t_{k-1}^i$  by at most  $(A_{k-1}^i + H_{k-1}^i)\alpha_k^i Q_{t_k^i}^i$ .

(iii) Moving  $(1 - \alpha_k^i)Q_{t_k^i}^i$  from period  $t_k^i$  to  $t_{k+1}^i$  results in an increase in cost at period  $t_{k+1}^i$  of at most  $A_{k+1}^i(1 - \alpha_k^i)Q_{t_k^i}^i \leq (A_k^i + H_k^i)(1 - \alpha_k^i)Q_{t_k^i}^i$ , where the inequality is true because the solution at any iteration satisfies the second property in Lemma 3.3, and a reduction in inventory costs of  $H_k^i(1 - \alpha_k^i)Q_{t_k^i}^i$ .  $\square$

LEMMA 3.5. *At each iteration of the combining step for a certain index  $k$  and retailer  $i$ , the increase in cost is no more than*

$$\frac{1}{3} \left[ (C_{k-1}^{*i} + H_{k-1}^{*i})(1 - \alpha_{k-1}^{*i})Q_{t_{k-1}^{*i}}^{*i} + C_k^{*i} \alpha_k^{*i} Q_{t_k^{*i}}^{*i} \right].$$

PROOF. The proof of this result uses the following key result in Chan et al. (2002, see proof of Lemma 4.3):

$$\min(\alpha A, \beta B) \leq \frac{1}{3}((\alpha + \beta)A + \alpha B) \quad \text{for all } \alpha, \beta, A, B \geq 0. \quad (3)$$

The increase in cost in the combine step associated with the  $k$ th order of retailer  $i$  is no more than

$$\begin{aligned} & \min \left\{ A_k^i(1 - \alpha_{k-1}^i)Q_{t_{k-1}^i}^i, (A_{k-1}^i + H_{k-1}^i - A_k^i)\alpha_k^i Q_{t_k^i}^i \right\} \\ & \leq \frac{1}{3} \left[ (A_{k-1}^i + H_{k-1}^i)(1 - \alpha_{k-1}^i)Q_{t_{k-1}^i}^i + A_k^i \alpha_k^i Q_{t_k^i}^i \right], \end{aligned}$$

where the inequality is a direct consequence of (3). Now, because  $K_{t_x^i}^0(\cdot)$  is concave and at each iteration  $Q_{t_x^i}^0 \geq \sum_{y=i}^n Q_{t_x^i}^y$  for  $x \geq k-1$ , we have that

$$K_{t_x^i}^0(Q_{t_x^i}^0) - K_{t_x^i}^0(Q_{t_x^i}^0 - Q_{t_x^i}^i) \leq K_{t_x^i}^0\left(\sum_{y=i}^n Q_{t_x^i}^y\right) - K_{t_x^i}^0\left(\sum_{y=i+1}^n Q_{t_x^i}^y\right).$$

Therefore, for all  $x \geq k-1$ ,

$$\begin{aligned} A_x^i & \equiv \frac{K_{t_x^i}^0(Q_{t_x^i}^0) - K_{t_x^i}^0(Q_{t_x^i}^0 - Q_{t_x^i}^i) + K_{t_x^i}^i(Q_{t_x^i}^i)}{Q_{t_x^i}^i} \\ & \leq \frac{K_{t_x^i}^0(\sum_{j=i}^n Q_{t_x^i}^j) - K_{t_x^i}^0(\sum_{j=i+1}^n Q_{t_x^i}^j) + K_{t_x^i}^i(Q_{t_x^i}^i)}{Q_{t_x^i}^i} \equiv C_x^i. \end{aligned}$$



Now observe that  $C_x^i \leq C_x^{*i}$  for all  $x \geq k-1$ , because  $t_x^i = t_x^{*i}$ ,  $Q_{t_x}^i = Q_{t_x}^{*i}$  for  $x > k-1$ , and  $Q_{t_{k-1}}^i \geq Q_{t_{k-1}}^{*i}$ .

Thus, the increase in cost is bounded by

$$\begin{aligned} & \frac{1}{3} \left[ (C_{k-1}^{*i} + H_{k-1}^i)(1 - \alpha_{k-1}^i) Q_{t_{k-1}}^i + C_k^{*i} \alpha_k^i Q_{t_k}^i \right] \\ &= \frac{1}{3} \left[ (C_{k-1}^{*i} + H_{k-1}^{*i})(1 - \alpha_{k-1}^{*i}) Q_{t_{k-1}}^{*i} + C_k^{*i} \alpha_k^{*i} Q_{t_k}^{*i} \right]. \end{aligned}$$

The last equality is due to the fact that the current solution  $S$ , before the iteration is performed, satisfies  $s_x^i = s_x^{*i}$ ,  $t_x^i = t_x^{*i}$ ,  $\alpha_x^i = \alpha_x^{*i}$ ,  $Q_{t_x}^i = Q_{t_x}^{*i}$  for  $x \geq k$ , and  $(1 - \alpha_{k-1}^i) Q_{t_{k-1}}^i = (1 - \alpha_{k-1}^{*i}) Q_{t_{k-1}}^{*i}$ .  $\square$

It remains to show that the bound is tight. Chan et al. (2002) show that there exist instances of the one-warehouse multiretailer problem with a single retailer for which the ratio  $Z^{ZIO}/Z^*$  is arbitrarily close to 4/3. Thus, the 4/3 bound cannot be improved in the case of multiple retailers.

Observe that the only properties of the modified all-unit discount function used in the proof of Theorem 2.2 are that it is nondecreasing in the quantity shipped and that the cost per unit is nonincreasing in that quantity. Hence, the theorem holds true for any LTL transportation function satisfying those properties. In a similar way, holding costs can be generalized to be any function of the quantity held that satisfies those two properties.

### 3.2. Stationary Ordering Cost Functions

In this case, we need to consider solutions satisfying conditions slightly different than those in the previous section. For this purpose, let

$$K^0(Q) = \lim_{\delta \rightarrow 0^+} \frac{K^0(Q) - K^0(Q - \delta)}{\delta}$$

and

$$B_k^i = K^0(Q_{t_k}^0) + \frac{K^i(Q_{t_k}^i)}{Q_{t_k}^i},$$

for each retailer  $i$ ,  $i = 1, 2, \dots, n$ , and ordering index  $k = 1, 2, \dots, m_i$ . Note that we have dropped the time subindex in the transportation cost functions because they are constant over time. That is,  $K_t^0(\cdot) = K^0(\cdot)$  and  $K_t^i(\cdot) = K^i(\cdot)$  for all  $i$  and  $t$ .

**OBSERVATION 3.6.**  $K^0(Q_{t_k}^0)$  is an upper bound on the per-unit cost that additional units to be delivered to the

warehouse at time  $t_k^i$  would incur and, at the same time, a lower bound on the cost per unit incurred by the current  $Q_{t_k}^0$  units being sent in that period.

This is due to the concavity of the warehouse ordering cost function,  $K^0(\cdot)$ .

**OBSERVATION 3.7.** Similarly,  $B_k^i$  is an upper bound on the per-unit transportation cost that additional units delivered to the retailer at time  $t_k^i$  would incur, and, at the same time, a lower bound on the cost per unit incurred by the current  $Q_{t_k}^0$  units being sent in that period.

This is explained by the previous observation and the fact that the cost per unit resulting from the LTL charges does not increase as the shipment becomes larger. We are now ready to introduce the counterpart of Lemma 3.3 for the stationary case.

**LEMMA 3.8.** Given any feasible policy  $S$ , there exists a feasible policy with lower or equal cost satisfying the following properties.

1. A positive fraction of any order is used to satisfy demand for periods previous to the arrival of the subsequent order. That is,  $t_{k+1}^i > s_k^i \geq t_k^i$  for all  $i = 1, 2, \dots, n$  and  $k = 1, 2, \dots, m_i$ .
2. If the order at  $t_k^i$  covers demands occurring after the next order at period  $t_{k+1}^i$  has arrived, then  $B_k^i + H_k^i > B_{k+1}^i$ .

**PROOF.** The proof is similar to that of Lemma 3.3. Please refer to the technical appendix for details ([mansci.pubs.informs.org/ecompanion.html](http://mansci.pubs.informs.org/ecompanion.html)).  $\square$

As in the previous section, we transform an optimal policy which satisfies the properties in Lemma 3.8 into a ZIO policy and show that the increase in cost due to the transformation is no more than 1/4.6 times the cost of the optimal solution. However, this case requires further detail to show that the tighter bound holds. In particular, one more way of combining orders needs to be introduced:

**Combine 3:** Move  $Q_{t_k}^i$  units from period  $t_k^i$  to period  $t_{k-1}^i$ , and Step 2 of the transformation procedure is replaced by the Step 2' described below. Let

$$K_1 \equiv H_{k-1}^i + K^0(Q_{t_{k-1}}^0) + \frac{K^i(Q_{t_{k-1}}^i)}{Q_{t_{k-1}}^i} \equiv H_{k-1}^i + B_{k-1}^i,$$

$$K_2 \equiv K^0(Q_{t_k}^0) + \frac{K^i(Q_{t_k}^i)}{Q_{t_k}^i} \equiv B_k^i,$$

$$H \equiv H_{k-1}^i + K^{0'}(Q_{i_{k-1}}^0) - K^{0'}(Q_{i_k}^0),$$

$$D \equiv (1 - \alpha_{k-1}^i)Q_{i_{k-1}}^i + \alpha_k^i Q_{i_k}^i,$$

$$\alpha \equiv \frac{(1 - \alpha_{k-1}^i)Q_{i_{k-1}}^i}{D},$$

$$\beta \equiv \frac{(1 - \alpha_k^i)Q_{i_k}^i}{D}.$$

- Step 2'. Combining step  $k$  at retailer  $i$ .
  - If 4.6  $\alpha(K_2 - H) \leq \alpha K_1 + (1 - \alpha)K_2$ , execute **combine 1**.
  - **Else if** 4.6  $(1 - \alpha)(K_1 - K_2) \leq \alpha K_1 + (1 - \alpha)K_2 + \beta K_2$ , execute **combine 2**.
  - **Else** execute **combine 3**.

Thus, in the remainder of this section we consider a transformation procedure identical to that in §3.1 except that Step 2 is replaced by Step 2' and the solution considered at each iteration satisfies the conditions in Lemma 3.8 rather than those in Lemma 3.3.

For any inventory ordering policy  $S$ , let  $Z(S)$  be the systemwide cost associated with policy  $S$  and  $Z(S, i, k)$  be the total cost associated with satisfying the demands of retailer  $i$  at periods  $t_k^i$  through  $T$  and all the demands (from period 1 to  $T$ ) of retailers  $i + 1$  through  $n$ . Let  $k$  be the index of the earliest ordering period at retailer  $i$  in which an order is placed before all inventory from the previous order has been depleted. Observe that to prove Theorem 2.3, it suffices to show that at any iteration of the transformation procedure the current on-hand solution  $S$  satisfies

$$Z(S) \leq Z(S, i, k) + \frac{5.6}{4.6}[Z^* - Z(S, i, k)]. \quad (4)$$

Obviously, this condition holds for  $S = S^*$ . The following lemmas show that if the current on-hand solution satisfies (4), then it also holds after executing Step 2' and Step 3. This implies that the current solution at any iteration of the transformation procedure satisfies (4). Finally, observe that the solution  $S$  generated at the last iteration satisfies  $Z(S, i, k) = 0$ , since it is a ZIO policy, and thus Theorem 2.3 follows directly from (4).

LEMMA 3.9. *If the current solution  $S$  satisfies (4), then it continues to hold after Step 3 has been executed.*

PROOF. Please refer to the technical appendix for details.  $\square$

LEMMA 3.10. *If the current solution  $S$  satisfies (4), then it continues to hold after Step 2' has been executed.*

PROOF. This is proven separately for each of the three possible combine steps. Please refer to the technical appendix for details.  $\square$

## 4. Optimal Zero-Inventory-Ordering Policy

In this section we show that when the number of retailers is fixed, we can find the best ZIO policy by formulating an associated shortest-path problem, in time which is polynomial in  $T$  and exponential in the number of retailers  $n$ . As we have seen, this ZIO policy has a cost within a factor of 4/3 from the optimal cost.

Let  $\mathcal{T} = \{1, 2, \dots, T + 1\}$  be the set of different time periods, where  $T + 1$  is used for notational convenience. Let  $N = \{1, 2, \dots, n\}$  be the set of retailers. Construct an acyclic graph  $G = (V, A)$ , where

$$\begin{aligned} V &= \{\bar{u} = \langle u_1, \dots, u_n \rangle \mid u_i \in \mathcal{T}, i = 1, \dots, n\} \\ &= \underbrace{\mathcal{T} \times \mathcal{T} \times \dots \times \mathcal{T}}_{n \text{ times}}, \end{aligned}$$

$$\begin{aligned} A &= \{ \langle u_1, \dots, u_n \rangle \rightarrow \langle v_1, \dots, v_n \rangle \mid \\ &\quad v_i \geq u_i \text{ for all } i; \\ &\quad \text{there is at least one component } i \\ &\quad \text{such that } u_i < v_i; \text{ for every } i \text{ with } u_i < v_i \\ &\quad \text{we have } u_i = \min_{\{j=1,2,\dots,n\}} u_j \equiv u \\ &\quad \text{(i.e., all the components that changed had} \\ &\quad \text{the same value, } u) \}. \end{aligned}$$

Given an arc  $\bar{u} \rightarrow \bar{v}$ , where  $\bar{u} = \langle u_1, \dots, u_n \rangle$  and  $\bar{v} = \langle v_1, \dots, v_n \rangle$ , let  $k$  be the number of components that are different in  $\bar{u}$  and  $\bar{v}$ , and  $I = \{i_1, i_2, \dots, i_k\}$  be the set of indices of those components; that is, for  $l = 1, 2, \dots, k$ ,  $i_l$  is such that  $u_{i_l} < v_{i_l}$ . Observe that  $k \geq 1$  and by construction  $u_{i_1} = u_{i_2} = \dots = u_{i_k} = u$ . The arc  $\bar{u} \rightarrow \bar{v}$  represents ordering at period  $u$  to satisfy demands of each retailer  $i_l, l = 1, 2, \dots, k$ , from period  $u$  through  $v_{i_l} - 1$ . Thus, the cost associated with this arc is the cost of ordering those units at period  $u$  and

holding them in inventory until their consumption. Specifically, the cost of this arc is

$$K_u^0(d_{uv_1}^{i_1} + d_{uv_2}^{i_2} + \dots + d_{uv_k}^{i_k}) + c_{uv_1}^{i_1} + c_{uv_2}^{i_2} + \dots + c_{uv_k}^{i_k},$$

where

- $d_{uv}^i$  is the total demand faced by retailer  $i$  from period  $u$  to  $v - 1$ ;
- $K_u^0(d_{uv_1}^{i_1} + d_{uv_2}^{i_2} + \dots + d_{uv_k}^{i_k})$  is the cost of shipping  $d_{uv_1}^{i_1} + d_{uv_2}^{i_2} + \dots + d_{uv_k}^{i_k}$  units from the supplier to the warehouse at time  $u$ ; and,
- $c_{uv}^i$  is the total shipping and holding cost for retailer  $i$  if we order at period  $u$  to cover the demands in periods  $u, u + 1, \dots, v - 1$ ; i.e.,

$$c_{uv}^i = K_u^i(d_{uv}^i) + \sum_{j=u}^{v-2} h_j^i d_{j+1v}^i.$$

It is easy to see that the shortest path from  $\langle 1, 1, \dots, 1 \rangle$  to  $\langle T + 1, T + 1, \dots, T + 1 \rangle$  in  $G = (V, A)$  corresponds to finding the best ZIO policy. Using a Fibonacci heap implementation of Dijkstra's algorithm (Cormen et al. 1999), the shortest path can be found in time  $O(|V| \log |V| + |A|)$ , where  $|V| = O((T + 1)^n)$ ,  $|A| = O((T)^{2n})$ , and  $n$  is the number of retailers. The algorithm grows to be computationally expensive as the number of retailers increases. Thus, the next step is to develop a heuristic that finds a good ZIO policy in polynomial time. This is the objective of the next section.

## 5. Linear-Programming-Based Algorithm

In this section we introduce a linear-programming-based heuristic that generates close-to-optimal ZIO policies and, thus, effective solutions to the single-warehouse multiretailer problem.

We start by formulating the problem of finding an optimal ZIO policy as an integer program. The algorithm is based on solving the linear programming relaxation of the resulting model and transforming the fractional solution obtained into an integer solution. Our method is similar to the method presented in Chan et al. (1999) for multicommodity network flows with piecewise linear concave costs.

The piecewise linear concave costs associated with shipments from the supplier to the warehouse are

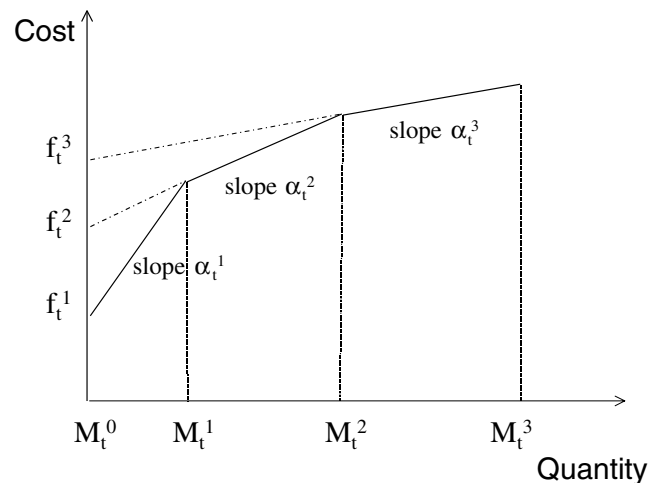
modeled as follows. Consider any time period  $t$  and the associated warehouse ordering cost function,  $K_t^0(\cdot)$ . Let  $R$  be the number of different slopes in the cost function, which we assume, without loss of generality, is the same for all time periods to avoid cumbersome notation. Let  $M_t^{r-1}, M_t^r$ , for  $r = 1, \dots, R$ , denote the lower and upper limits, respectively, on the interval corresponding to the  $r$ th slope of the piecewise linear cost function. Note that  $M_t^0 = 0$  and  $M_t^R$  can be set to the total quantity that may be shipped at period  $t$ ,  $\sum_{i \geq t} \sum_{j=1}^n d_j^i$ . We associate with each of these intervals, say  $r$ , a variable cost per unit, denoted by  $\alpha_t^r$ , equal to the slope of the corresponding line segment, and a fixed cost,  $f_t^r$ , defined as the y-intercept of the linear prolongation of that segment. See Figure 5 for a graphical representation.

Observe that the cost incurred by shipping a quantity on a certain range is the sum of its associated fixed cost plus the cost of shipping all units at its corresponding linear cost. That is, if we let  $Q_t^0$  denote the warehouse order at time  $t$ , we can express the associated transportation cost,  $K_t^0(Q_t^0)$ , as  $K_t^0(Q_t^0) = f_t^r + \alpha_t^r Q_t^0$ , where  $r$  is such that  $Q_t^0 \in (M_t^{r-1}, M_t^r]$ .

Finally, we define the following variables. For each  $t = 1, 2, \dots, T$  and  $r = 1, 2, \dots, R$ , let

$$U_t^r = \begin{cases} 1 & \text{if } Q_t^0 \in (M_t^{r-1}, M_t^r], \\ 0 & \text{otherwise.} \end{cases}$$

Figure 5 Piecewise Linear Concave Function



For each retailer  $i = 1, 2, \dots, n$ , and periods  $1 \leq t \leq k \leq T$ , let  $Z_{tk}^i =$  quantity ordered by retailer  $i$  at time  $t$  to satisfy demand at period  $k \geq t$  and

$$Z_{tk}^{ir} = \begin{cases} Z_{tk}^i & \text{if } Q_t^0 \in (M_t^{r-1}, M_t^r], \\ 0 & \text{otherwise,} \end{cases}$$

for each  $r = 1, 2, \dots, R$ . In what follows we refer to the  $U$  variables as *interval variables* and to the  $Z$  variables as *quantity variables*.

To model ordering and inventory costs at the retailer level, we consider a dummy period  $T+1$  and define, as in §4, for each retailer  $i = 1, 2, \dots, n$  and periods  $1 \leq t < k \leq T+1$ ,  $c_{tk}^i =$  total cost of ordering at period  $t$  to satisfy demand for periods  $t$  through  $k-1$  and holding the units in inventory until their consumption. That is,

$$c_{tk}^i = K_t^i \left( \sum_{j=t}^{k-1} d_j^i \right) + \sum_{j=t}^{k-2} h_j^i \left( \sum_{l=j+1}^{k-1} d_l^i \right).$$

Observe that a ZIO policy for retailer  $i$  can be interpreted as a path from 1 to  $T+1$  on a network with nodes  $\{1, 2, \dots, T+1\}$  and arcs  $(t, k)$ , for  $1 \leq t < k \leq T+1$ , with associated cost  $c_{tk}^i$ . In what follows, we refer to this network as the  *$i$ th retailer's network* or  $G_i$ .

Thus, to calculate ordering and inventory costs at retailer  $i$  we formulate a shortest-path model on  $G_i$  using the variables

$$X_{tk}^i = \begin{cases} 1 & \text{if an order is placed by retailer } i \text{ at time } t \\ & \text{to satisfy demands for periods } t \\ & \text{through } k-1, \\ 0 & \text{otherwise,} \end{cases}$$

and flow conservation constraints. We refer to  $X = (X_{tk}^i)$  as the *vector of path flows*.

The best ZIO policy can be found by solving the following integer program:

Problem  $P$ :

$$\begin{aligned} \text{Min} \quad & \sum_{t=1}^T \sum_{r=1}^R \left[ f_t^r U_t^r + \alpha_t^r \sum_{i=1}^n \sum_{k=t}^T Z_{tk}^{ri} \right] \\ & + \sum_{i=1}^n \sum_{t=1}^T \sum_{k=t+1}^{T+1} c_{tk}^i X_{tk}^i \end{aligned}$$

$$\text{s.t.} \quad Z_{tk}^{ri} \leq d_k^i U_t^r \quad \forall r = 1, 2, \dots, R, \\ i = 1, 2, \dots, n, \text{ and } 1 \leq t \leq k \leq T, \quad (5)$$

$$\sum_{r=1}^R Z_{tk}^{ri} = d_k^i \sum_{l=k+1}^{T+1} X_{tl}^i \quad \forall 1 \leq t \leq k \leq T, \\ i = 1, 2, \dots, n, \quad (6)$$

$$\sum_{j:j>l} X_{lj}^i - \sum_{j:j<l} X_{jl}^i = \begin{cases} 1 & \text{if } l = 1 \\ -1 & \text{if } l = T+1 \\ 0 & \text{if } 1 < l \leq T \end{cases} \quad (7)$$

$$\forall i = 1, 2, \dots, n,$$

$$Z_{tk}^{ri} \geq 0 \quad \forall r = 1, 2, \dots, R, \quad i = 1, 2, \dots, n \\ \text{and } 1 \leq t \leq k \leq T,$$

$$U_t^r \in \{0, 1\} \quad \forall r = 1, 2, \dots, R \\ \text{and } t = 1, 2, \dots, T,$$

$$X_{tk}^i \in \{0, 1\} \quad \forall i = 1, 2, \dots, n \\ \text{and } 1 \leq t \leq k \leq T+1.$$

The first set of constraints, (5), specifies that if some quantity is ordered at time  $t$  by any retailer and shipped on interval  $r$  of the transportation cost function, then the associated interval variable,  $U_t^r$ , must be 1. These, together with the integrality of the  $U$  variables, are the only constraints needed to model the piecewise linear concave costs; see Chan et al. (1999). Obviously, constraints (5) could be aggregated for all  $k$ . However, this would considerably weaken the linear programming relaxation of Problem  $P$ . Equation (6) guarantees that if a positive amount is shipped to retailer  $i$  at time  $t$  to satisfy demand at the retailer at period  $k$ , then the retailer must order at period  $t$  to cover demands for periods  $t$  through some  $l-1 \geq k$ . Observe that these constraints link the supplier-warehouse model with the retailer model. Finally, the flow conservation constraints (7) correspond to finding, for each retailer  $i$ , a path from 1 to  $T+1$  on the retailer's network,  $G_i$ .

Unfortunately, solving this integer program is computationally intractable for all but small-size problems. To overcome this difficulty, we focus on analyzing the behavior of its linear programming relaxation and take advantage of its structural properties to develop an effective heuristic that constructs

an integer solution to Problem  $P$  from its optimal fractional solution.

For that purpose, we start by fixing the fractional path flows and studying the behavior of the resulting linear program. Let  $X = (X_{ik}^i)$  be the vector of path flows in a feasible solution to the linear programming relaxation of Problem  $P$ . What is the cost that the linear program associates with this solution? What are the values of the corresponding interval and quantity variables,  $U_t^r$  and  $Z_{ik}^{ri}$ ?

Observe that given the vector of fractional path flows  $X$ , the amount to be shipped from the supplier to the warehouse at any time period is known and therefore the linear programming relaxation of Problem  $P$  can be decomposed into multiple subproblems, one for every time period. For each time period  $t$ , the total shipping cost, as well as the corresponding variables  $Z_{ik}^{ri}$  and  $U_t^r$ , can be obtained by solving the following problem, which we refer to as the fixed-flow subproblem at period  $t$  or Problem  $FF_X^t$ :

$$\begin{aligned} \text{Problem } FF_X^t: \quad & \text{Min} \quad \sum_{r=1}^R \left[ f_t^r U_t^r + \alpha_t^r \sum_{i=1}^n \sum_{k=t}^T Z_{ik}^{ri} \right] \\ \text{s.t.} \quad & Z_{ik}^{ri} \leq d_k^i U_t^r \quad \forall k = t, \dots, T \\ & \text{and } r = 1, \dots, R, \end{aligned} \tag{8}$$

$$\begin{aligned} \sum_{r=1}^R Z_{ik}^{ri} &= d_k^i \sum_{s=k+1}^{T+1} X_{ts}^i \quad \forall k = t, \dots, T \\ & \text{and } i = 1, \dots, n, \end{aligned} \tag{9}$$

$$\begin{aligned} Z_{ik}^{ri} &\geq 0 \quad \forall k = t, \dots, T, \quad i = 1, \dots, n, \\ & \text{and } r = 1, \dots, R, \end{aligned}$$

$$U_t^r \geq 0 \quad \forall r = 1, \dots, R.$$

The following lemma, presented in Chan et al. (1999), explicitly characterizes the solution to the linear program for any given (fixed) vector of path flows  $X$ .

**LEMMA 5.1.** *For any given period of time  $t$  and fixed vector of path flows  $X$ , let the proportion of the demand of retailer  $i$  at time  $k$  which is shipped at period  $t$  be*

$$\gamma_{ik}^i \equiv \sum_{s=k+1}^{T+1} X_{ts}^i,$$

for  $k \geq t$  and  $i = 1, 2, \dots, n$ . Rank the proportions,  $\gamma_{ik}^i$ , in nondecreasing order of their values. Following that order, associate a single index  $l$  to each pair  $(i, k)$  so that  $\gamma_{il} \equiv \gamma_{ik}^i$  and

$$\gamma_{l1} \leq \gamma_{l2} \leq \dots \leq \gamma_{lL},$$

where  $L = n \times (T - t + 1)$ . Similarly, let  $d_l \equiv d_k^i$  for each  $l = 1, 2, \dots, L$  and its corresponding pair  $(i, k)$ . Then, the optimal solution to the fixed-flow subproblem at time  $t$  is

$$\sum_{l=1}^L K_t^0 \left( \sum_{s=l}^L d_s \right) [\gamma_{tl} - \gamma_{t,l-1}] \quad \text{where } \gamma_{t0} := 0. \tag{10}$$

We are now ready to describe a polynomial time heuristic that finds an effective ZIO policy based on the solution to the linear programming relaxation of Problem  $P$ . Lemma 5.1 will be extensively used by the algorithm to compute the increase in costs in the solution to the linear program when the vector  $X$  is modified in the search for an integer solution.

### Linear-Programming-Based Algorithm

*Step 1.* Solve the linear programming relaxation of Problem  $P$ . Let  $X^* = (X_{ik}^{i*})$  be the optimal solution. Initialize  $i = 1$ .

*Step 2.* For each arc  $t \rightarrow k$ ,  $1 \leq t < k \leq T + 1$ , in network  $G_i$  compute a *marginal cost*,  $c_{tk}^i$ , as follows. The marginal cost is the total increase in cost in the solution to the linear program incurred when augmenting the flow on that arc from the fractional  $X_{tk}^{i*}$  to 1. That is,

$$c_{tk}^i = W_{tk}^i + (1 - X_{t,k}^{i*}) \cdot c_{t,k}^i,$$

where  $W_{tk}^i$  is the increase in transportation cost to the warehouse resulting from modifying flow in the linear program from  $X_{tk}^{i*}$  to 1. This cost increase can be easily calculated using Lemma 5.1.

*Step 3.* Determine the ordering epochs of retailer  $i$  by finding the minimum cost path from 1 to  $T + 1$  on network  $G_i$  with edge costs equal to the marginal costs.

*Step 4.* Update the amount and costs of warehouse orders at each period to account for retailer  $i$ 's ordering strategy. Costs are updated using Lemma 5.1.

*Step 5.* Let  $i = i + 1$  and repeat Steps (2)–(5) until  $i = n + 1$ .

This algorithm runs in polynomial time. It requires solving a linear program with  $O(T^2Rn)$  variables and  $O(T^2Rn)$  constraints, and then performing Steps 2 through 5 to obtain an integer solution. Observe that when applying Lemma 5.1 at any step of the algorithm, the ranking of the proportions associated with the flow on an edge  $t$  from supplier to warehouse can be updated easily in  $O(nT)$  because proportions are only changed to values of 0 or 1. The transportation cost on edge  $t$  can then be computed in  $O(nT + R)$ . This is done at most  $O(nT^2)$  times over all iterations of Steps 2 and 4. Consequently, the complexity of performing Steps 2 through 5 is no more than  $O(n^2T^3 + nT^2R)$ .

## 6. Computational Results

In this section we test the performance of the linear-programming-based algorithm in terms of both computational time and relative deviation from the optimal ZIO policy. For this purpose, the linear-programming-based algorithm was coded in C++ and a variety of test problems, described below, were solved on a Sun Ultra 1 SPARCstation; CPLEX 5.0 was used to solve the linear and integer programming problems.

We apply the algorithm to two types of problems. The first is the single-warehouse multiretailer problem with retailer ordering cost represented by the modified all-unit discount cost function. The second is the single-warehouse multiretailer problem with concave ordering cost functions for the retailers. Of course, in both types of problems, warehouse ordering cost is a piecewise linear concave function. Observe that there exists an optimal ZIO policy for the second type of problems.

**Type 1 Instances.** We consider four problem classes corresponding to 5, 25, 50, and 100 retailers. The planning horizon is 12 periods and demands for each retailer are generated from a normal distribution with mean 100 and standard deviation 20. Holding costs are randomly generated in the interval  $[0.1, 0.6]$ . Supplier-warehouse transportation costs are described by piecewise linear concave functions with five breakpoints between 0 and the maximum amount that could possibly be ordered from the

supplier to satisfy retailer demands. We consider the breakpoints fixed and randomly vary fixed costs and slopes (within a certain sensible range) over time. Similarly, the warehouse-retailer transportation cost function for a particular retailer is a modified all-unit discount function with either 6, 7, or 8 price breaks and again fixed costs and slopes are randomly varied over time.

Table 1 shows, for each problem class, the average computation time of the linear-programming-based algorithm over 200 instances generated. For these moderate-size instances tested, the optimal ZIO can be calculated by solving the integer program, Problem  $P$ , and used to evaluate the performance of the heuristic solution. The associated average computation times are given in the fourth column of Table 1. The cost of the optimal ZIO policy obtained is compared to the heuristic solution in the last two columns: The first column reports the average ratio for cases in which the solution to the linear programming relaxation of Problem  $P$  was not integer. The second reports the average over all problems tested.

**Type 2 Instances.** Here we study the performance of the linear-programming-based algorithm for instances in which all the transportation costs are piecewise linear and concave.

We again consider different problem classes, with normally and independently, identically distributed retailer demands with mean 100 and standard deviation 20, and generate 10 instances for each class. Holding costs are set to 0.2 per unit per period. The piecewise linear concave transportation costs considered have three price breaks (i.e., four segments with different slope) in the range from 0 to the maximum possible demand that could be satisfied using that link. Associated fixed costs and variable costs are randomly generated over time.

Table 2 describes the six problem classes tested and reports the average computation time and the average ratio of heuristic to optimal solutions over the five instances tested for each class. We observe that in all the instances tested, the solution to the linear programming relaxation coincides with the optimal integer solution.

**Table 1** Computational Results for Modified All-Unit Discount Warehouse-Retailer Costs

Problem class	Number of retailers	CPU time (seconds)	CPU Time with IP (seconds)	Frequency of fractional solution	$Z^H/Z^{ZIO}$ Fractional cases	$Z^H/Z^{ZIO}$ All cases
Class 1	5	$\approx 0$	3	4/200	1.010	1.0002
Class 2	25	$\approx 0$	27	5/200	1.013	1.0004
Class 3	50	2	124	3/200	1.037	1.0006
Class 4	100	23	507	2/200	1.025	1.0003

**Table 2** Computational Results for Concave Transportation Costs on All Links

Problem class	Number of periods	Number of retailers	CPU Time (seconds)	$Z^H/Z^{ZIO} = Z^H/Z^*$
Class 1	6	5	$\approx 0$	1
Class 2	12	5	1	1
Class 3	6	10	$\approx 0$	1
Class 4	12	10	4	1
Class 5	6	25	2	1
Class 6	12	25	5	1

## 7. Conclusions and Extensions

We should point out that the worst-case results described in this paper hold under fairly general settings. Indeed, a careful inspection of the proofs of Theorems 2.2 and 2.3 reveals that these results hold for any general warehouse-retailer ordering functions and retailer holding costs satisfying the following properties: (i) they are nondecreasing functions of the amounts shipped and held, respectively, and (ii) the associated cost per unit is nonincreasing in those amounts. Moreover, both conditions are necessary to obtain a finite worst-case performance bound on the best ZIO policy. That is, if at least one of these conditions does not hold, then the best ZIO policy has an unbounded worst-case performance (see Chan et al. 2002).

Finally, the bounds on the performance of ZIO policies developed in this paper can be easily extended to a more general distribution problem with central stocks, in which the warehouse is allowed to carry

inventory. For a proof, please refer to the technical appendix to this paper available on the *Management Science* electronic companion page at <http://mansci.pubs.informs.org/ecompanion./html>.

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