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INTEGER PROGRAMMING AND ARROVIAN SOCIAL WELFARE FUNCTIONS

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We characterize the class of Arrovian Social Welfare Functions (ASWFs) as integer solutions to a collection of linear inequalities. Many of the classical possibility, impossibility, and characterization results can be derived in a simple and unified way from this integer program. Among the new results we derive is a characterization of preference domains that admit a nondictatorial, neutral ASWF. We also give a polyhedral characterization of all ASWFs on single-peaked domains.

1. Introduction. The Old Testament likens the generations of men to the leaves of a tree. It is a simile that applies as aptly to the literature inspired by Arrow's Impossibility Theorem (Arrow 1963). Much of it is devoted to classifying those preference domains that admit or exclude the existence of a nondictatorial Arrovian Social Welfare Function (ASWF). (An ASWF is a social welfare function that satisfies the axioms of the Impossibility Theorem.) We add another leaf to that tree. Here we characterize the class of ASWFs as solutions to an integer program. This formulation allows us to derive in a systematic way many of the known results about ASWFs. It is inspired by a characterization of Arrovian domains due to Kalai and Muller (1977).

Let $\mathcal{A}$ denote the set of alternatives (at least three). Let Σ denote the set of all transitive, antisymmetric, and total binary relations on $\mathcal{A}$. An element of Σ is a preference ordering. Notice that this setup excludes indifference. The set of admissible preference orderings for a society of n-agents (voters) will be a subset of Σ and denoted Ω. Let Ω^n be the set of all n-tuples of preferences from Ω, called *profiles*. An element of Ω^n will typically be denoted as $\mathbf{P} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n)$, where $\mathbf{p}_i$ is interpreted as the preference ordering of agent i. In the language of Le Breton and Weymark (1996), we assume the *common preference domain* framework.

An n-person social welfare function is a function $f\colon \Omega^n \to \Sigma$. Thus, for any $\mathbf{P} \in \Omega^n$, $f(\mathbf{P})$ is an ordering of the alternatives. We write $x\, f(\mathbf{P})\, y$ if x is ranked above y under $f(\mathbf{P})$. An n-person ASWF on Ω is a function $f\colon \Omega^n \to \Sigma$ that satisfies the following two conditions:

1. **Unanimity.** If for $\mathbf{P} \in \Omega^n$ and some $x, y \in \mathcal{A}$, we have $x\, \mathbf{p}_i\, y$ for all i, then $x\, f(\mathbf{P})\, y$.
2. **Independence of irrelevant alternatives.** For any $x, y \in \mathcal{A}$ suppose $\exists\, \mathbf{P}, \mathbf{Q} \in \Omega^n$ such that $x\, \mathbf{p}_i\, y$ if an only if $x\, \mathbf{q}_i\, y$ for $i = 1, \ldots, n$. Then, $x\, f(\mathbf{P})\, y$ if and only if $x\, f(\mathbf{Q})\, y$.

The first axiom stipulates that if all voters prefer alternative x to alternative y, then the social welfare function f must rank x above y. The second axiom states that the ranking of x and y in f is not affected by how the voters rank the other alternatives. An obvious social welfare function that satisfies the two conditions is the *dictatorial rule*: rank the alternatives in the order of the preferences of a particular voter (the dictator). Formally, an ASWF is *dictatorial* if there is an i such that $f(\mathbf{P}) = \mathbf{p}_i$ for all $\mathbf{P} \in \Omega^n$. An ordered pair $x, y \in \mathcal{A}$ is called *trivial* if $x\, \mathbf{p}\, y$ for all $\mathbf{p} \in \Omega$. In view of unanimity, any ASWF must have $x\, f(\mathbf{P})\, y$ for all $\mathbf{P} \in \Omega^n$ whenever x, y is a trivial pair. If Ω consists only of trivial pairs, then distinguishing between dictatorial and nondictatorial ASWFs becomes nonsensical, so

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we assume that Ω contains at least one nontrivial pair. The domain Ω is *Arrovian* if it admits a nondictatorial ASWF.

The conditions identified by Kalai and Muller (1977) for the existence of a 2-person nondictatorial ASWF have a natural interpretation as an integer programming problem. In fact, all 2-person ASWFs are solutions to this integer program. A natural question that arises is whether there exists a characterization of all n-person ASWFs. In this paper, we answer this question by observing that the axioms for ASWFs induce a natural integer programming formulation. This approach, while intuitive, allows us to derive several structural and impossibility theorems in social choice theory in a unified and simple way.

Contributions. The main contributions of this paper are summarized below.

- An integer linear programming formulation of the problem of finding an n-person ASWF. For each Ω we construct a set of linear inequalities with the property that every feasible 0–1 solution corresponds to an n-person ASWF. The formulation is an extension of the conditions identified by Kalai and Muller (1977) for the case $n = 2$ to general n. The characterization extends easily to the case when the common preference domain assumption is dropped.
- When restricted to the class of *neutral* ASWFs the integer program yields a simple and easily checkable characterization of domains that admit neutral, nondictatorial ASWFs.
- When Ω is single peaked, we show that the polytope defined by the set of linear inequalities is *integral*: the vertices of the polytope correspond to ASWFs and every ASWF corresponds to a vertex of the polytope. This gives the first characterization of ASWFs on this domain we are aware of. The same proof technique yields a characterization of the generalized majority rule on single-peaked domains, originally due to Moulin (1984).
- To illustrate the versatility of the integer program, we use it to derive dictatorship results for social choice functions under monotonicity (Muller and Satterthwaite 1977).
- In a different vein, we point out that the computational complexity of deciding whether a domain is Arrovian depends critically on the way the domain is described. In fact, for some domain descriptions, the integer programming formulation implied by Kalai and Muller (1977) cannot even be explicitly determined in polynomial time (unless $P = NP$), let alone checking the existence of a nontrivial integral solution.

Structure of this paper. The rest of this paper is organized as follows: In §2, we describe the integer programming (IP) formulation, and derive some useful consequences, including a short proof of Arrow's theorem. Section 3 considers social welfare functions with additional restrictions such as anonymity, neutrality, and monotonicity; the main result of this section is an efficient characterization of neutral ASWFs. The next two sections illustrate the power of the (IP): §4 considers single-peaked domains, and characterizes all ASWFs as extreme points of a linear program, and §5 applies the IP approach to social choice problems. Complexity issues are discussed in §6, followed by suggestions for future research in §7.

2. The integer program.

Denote the set of all ordered pairs of alternatives by $\mathcal{A}^2$. Let E denote the set of all agents, and S^c denote $E \setminus S$ for all $S \subseteq E$.

To construct an n-person ASWF, we exploit the independence of the irrelevant alternatives condition. This allows us to specify an ASWF in terms of which ordered pair of alternatives a particular subset, S, of agents is decisive over.

DEFINITION 1. For a given ASWF f, a subset S of agents is *weakly decisive for x over y* if whenever all agents in S rank x over y and all agents in S^c rank y over x, the ASWF f ranks x over y.

Because this is the only notion of decisiveness used in the paper, we omit the qualifier "weak" in what follows.

For each nontrivial element $(x, y) \in \mathcal{A}^2$, we define a 0–1 variable as follows:

$$d_S(x, y) = \begin{cases} 1 & \text{if the subset } S \text{ of agents is decisive for } x \text{ over } y, \\ 0 & \text{otherwise.} \end{cases}$$

We can extend this definition to trivial pairs in a natural way: If $(x, y) \in \mathcal{A}^2$ is a trivial pair then, by convention, we set $d_S(x, y) = 1$ for all $S \neq \varnothing$.

Given an ASWF f, we can determine the associated d variables as follows: For each $S \subseteq E$, and each nontrivial pair (x, y), pick a $\mathbf{P} \in \Omega^n$ in which agents in S rank x over y, and agents in S^c rank y over x; if $x\, f(\mathbf{P})\, y$, set $d_S(x, y) = 1$, else set $d_S(x, y) = 0$.

In the rest of this section, we identify conditions satisfied by the d variables associated with an ASWF f.

Unanimity. To ensure unanimity, for all $(x, y) \in \mathcal{A}^2$, we require

(1) $$d_E(x, y) = 1.$$

Independence of irrelevant alternatives. Consider a pair of alternatives $(x, y) \in \mathcal{A}^2$, $\mathbf{P} \in \Omega^n$, and let S be the set of agents that prefer x to y in $\mathbf{P}$. (Thus, each agent in S^c prefers y to x in $\mathbf{P}$.) Suppose $x\, f(\mathbf{P})\, y$. Let $\mathbf{Q} \in \Omega^n$ be any other profile such that all agents in S rank x over y and all agents in S^c rank y over x. By the independence of irrelevant alternatives condition $x\, f(\mathbf{Q})\, y$. Hence, the set S is decisive for x over y. However, if $y\, f(\mathbf{P})\, x$ a similar argument would imply that S^c is decisive for y over x. Thus, for all S and nontrivial $(x, y) \in \mathcal{A}^2$, we must have

(2) $$d_S(x, y) + d_{S^c}(y, x) = 1.$$

A consequence of Equations (1) and (2) is that $d_{\varnothing}(x, y) = 0$ for all $(x, y) \in \mathcal{A}^2$.

Transitivity. To motivate the next class of constraints, it is useful to consider the majority rule. If the number n of agents is odd, the majority rule can be described using the following variables:

$$d_S(x, y) = \begin{cases} 1 & \text{if } |S| > n/2, \\ 0 & \text{otherwise.} \end{cases}$$

These variables satisfy both (1) and (2). However, if Ω admits a *Condorcet* triple (e.g., $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \in \Omega$ with $x\, \mathbf{p}_1\, y\, \mathbf{p}_1\, z$, $y\, \mathbf{p}_2\, z\, \mathbf{p}_2\, x$, and $z\, \mathbf{p}_3\, x\, \mathbf{p}_3\, y$), then such a rule does not always produce an *ordering* of the alternatives for each preference profile. Our next constraint (*cycle elimination*) is designed to exclude this and similar possibilities.

Let A, B, C, U, V, and W be (possibly empty) *disjoint* sets of agents whose union includes all agents. For each such partition of the agents, and any triple x, y, z,

(3) $$d_{A\cup U\cup V}(x, y) + d_{B\cup U\cup W}(y, z) + d_{C\cup V\cup W}(z, x) \leq 2,$$

where the sets satisfy the following conditions (hereafter referred to as conditions (*); see Figure 1):

$A \neq \varnothing$ only if there exists $\mathbf{p} \in \Omega$, $x\, \mathbf{p}\, z\, \mathbf{p}\, y$,

$B \neq \varnothing$ only if there exists $\mathbf{p} \in \Omega$, $y\, \mathbf{p}\, x\, \mathbf{p}\, z$,

$C \neq \varnothing$ only if there exists $\mathbf{p} \in \Omega$, $z\, \mathbf{p}\, y\, \mathbf{p}\, x$,

$U \neq \varnothing$ only if there exists $\mathbf{p} \in \Omega$, $x\, \mathbf{p}\, y\, \mathbf{p}\, z$,

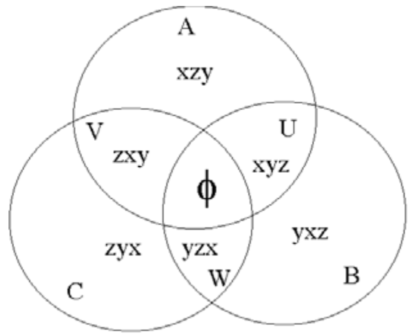

FIGURE 1. The sets and the associated orderings.

$$V \neq \varnothing \text{ only if there exists } \mathbf{p} \in \Omega, z \,\mathbf{p}\, x \,\mathbf{p}\, y,$$

$$W \neq \varnothing \text{ only if there exists } \mathbf{p} \in \Omega, y \,\mathbf{p}\, z \,\mathbf{p}\, x.$$

The constraint ensures that on any profile $\mathbf{P} \in \Omega^n$, the ASWF f does not produce a ranking that "cycles".

A useful consequence of (2) and (3) is

$$d_{A\cup U\cup V}(x, y) + d_{B\cup U\cup W}(y, z) + d_{C\cup V\cup W}(z, x) \geq 1. \tag{4}$$

To deduce it, interchange the roles of z and x in (3). Then, the roles of A and V (resp. B and W, C and U) can be interchanged to obtain the new inequality

$$d_{A\cup C\cup V}(z, y) + d_{B\cup C\cup W}(y, x) + d_{A\cup B\cup U}(x, z) \leq 2.$$

Using (2), we obtain

$$d_{B\cup U\cup W}(y, z) + d_{A\cup U\cup V}(x, y) + d_{C\cup V\cup W}(z, x) \geq 1.$$

Subsequently, we prove that constraints (1)–(3) are both necessary and sufficient for the characterization of n-person ASWFs. Before that, it is useful to develop a better understanding of constraint (3), and their relationship to the constraints identified in Kalai and Muller (1977) called *decisiveness implications*, described below.

Suppose there are $\mathbf{p}, \mathbf{q} \in \Omega$ and three alternatives x, y, and z such that $x \,\mathbf{p}\, y \,\mathbf{p}\, z$ and $y \,\mathbf{q}\, z \,\mathbf{q}\, x$. Kalai and Muller showed that $d_S(x, y) = 1 \Rightarrow d_S(x, z) = 1$, and $d_S(z, x) = 1 \Rightarrow d_S(y, x) = 1$. These conditions can be formulated as the following two inequalities:

$$d_S(x, y) \leq d_S(x, z), \tag{5}$$

$$d_S(z, x) \leq d_S(y, x). \tag{6}$$

The first condition follows from using a profile $\mathbf{P}$ in which agents in S rank x over y over z and agents in S^c rank y over z over x. If S is decisive for x over y, then $x\, f(\mathbf{P})\, y$. By unanimity, $y\, f(\mathbf{P})\, z$. By transitivity, $x\, f(\mathbf{P})\, z$. Hence, S is also decisive for x over z. The second condition follows from a similar argument.

CLAIM 1. Constraints (5) and (6) are special cases of constraint (3).

PROOF. Let $U \leftarrow S, W \leftarrow S^c$ in constraint (3), with the other sets being empty. U and W can be assumed nonempty by condition (*). Constraint (3) reduces to $d_U(x, y) + d_{U\cup W}(y, z) + d_W(z, x) \leq 2$. Because $U \cup W = E$, the above reduces to $0 \leq d_S(x, y) + d_{S^c}(z, x) \leq 1$, which implies $d_S(x, y) \leq d_S(x, z)$ by (2). By interchanging the roles of S and S^c, we obtain the inequality $d_{S^c}(x, y) \leq d_{S^c}(x, z)$, which is equivalent to $d_S(z, x) \leq d_S(y, x)$. □

Suppose we know only that there is a $\mathbf{p} \in \Omega$ with $x\,\mathbf{p}\,y\,\mathbf{p}\,z$. In this instance, transitivity requires $d_S(x, y) = 1$ and $d_S(y, z) = 1 \Rightarrow d_S(x, z) = 1$, and

$$d_S(z, x) = 1 \quad \Rightarrow \quad \text{at least one of } d_S(y, x) = 1 \text{ or } d_S(z, y) = 1.$$

These can be formulated as the following two inequalities:

$$d_S(x, y) + d_S(y, z) \leq 1 + d_S(x, z), \tag{7}$$

$$d_S(z, y) + d_S(y, x) \geq d_S(z, x). \tag{8}$$

Similarly, we have:

Claim 2. Constraints (7) and (8) are special cases of constraint (3).

Proof. Suppose $\exists\, \mathbf{q} \in \Omega$ with $x\,\mathbf{q}\,y\,\mathbf{q}\,z$. If there exists $\mathbf{p} \in \Omega$ with $y\,\mathbf{p}\,z\,\mathbf{p}\,x$ or $z\,\mathbf{p}\,x\,\mathbf{p}\,y$, then constraints (7) and (8) are implied by constraint (5) and 6), which, in turn, are special cases of constraint (3). So we may assume that there does not exist $\mathbf{p} \in \Omega$ with $y\,\mathbf{p}\,z\,\mathbf{p}\,x$ or $z\,\mathbf{p}\,x\,\mathbf{p}\,y$. If there does not exist $\mathbf{p} \in \Omega$ with $z\,\mathbf{p}\,y\,\mathbf{p}\,x$, then x, z is a trivial pair, and constraints (7) and (8) are redundant. So we may assume that such a $\mathbf{p}$ exists, hence, C can be chosen to be nonempty.

Let $U \leftarrow S, C \leftarrow S^c$ in constraint (3), with the other sets being empty. Constraint (3) reduces to

$$d_U(x, y) + d_U(y, z) + d_C(z, x) \leq 2,$$

which is just $d_S(x, y) + d_S(y, z) + d_{S^c}(z, x) \leq 2$.

Thus, constraint (7) follows as a special case of constraint (3). By reversing the roles of S and S^c again, we can show that constraint (8) follows as a special case of constraint (3). □

For $n = 2$, we can show that constraints (1)–(3) reduce to constraints (1), (2), and (5)–(8). Thus, constraint (3) generalize the decisiveness implication conditions to $n \geq 3$. We will sometimes refer to (1)–(3) as IP.

Theorem 1. *Every feasible integer solution to* (1)–(3) *corresponds to an ASWF and vice-versa.*

Proof. Given an ASWF, it is easy to see that the corresponding d vector satisfies (1)–(3). Now pick any feasible solution to (1)–(3) and call it d.

To prove that d gives rise to an ASWF, we show that for every profile of preferences from Ω, d generates an ordering of the alternatives. Unanimity and independence of irrelevant alternatives follow automatically from the way the d_S variables are used to construct the ordering.

Suppose d does not produce an ordering of the alternatives. Then, for some profile $\mathbf{P} \in \Omega^n$, there are three alternatives x, y, and z such that d ranks x over y, y over z, and z over x. For this to happen there must be three nonempty sets H, I, and J such that

$$d_H(x, y) = 1, \quad d_I(y, z) = 1, \quad d_J(z, x) = 1,$$

and for the profile $\mathbf{P}$, agent i ranks x over y (resp. y over z, z over x) if and only if i is in H (resp. I, J). Note that $H \cup I \cup J$ is the set of all agents, and $H \cap I \cap J = \varnothing$.

Let

$$A \leftarrow H \setminus (I \cup J), \quad B \leftarrow I \setminus (H \cup J), \quad C \leftarrow J \setminus (H \cup I),$$
$$U \leftarrow H \cap I, \quad V \leftarrow H \cap J, \quad W \leftarrow I \cap J.$$

Now A (resp. B, C, U, V, W) can only be nonempty if there exists $\mathbf{p}$ in Ω with $x\,\mathbf{p}\,z\,\mathbf{p}\,y$ (resp. $y\,\mathbf{p}\,x\,\mathbf{p}\,z$, $z\,\mathbf{p}\,y\,\mathbf{p}\,x$, $x\,\mathbf{p}\,y\,\mathbf{p}\,z$, $z\,\mathbf{p}\,x\,\mathbf{p}\,y$, $y\,\mathbf{p}\,z\,\mathbf{p}\,x$).

In this case constraint (3) is violated because

$$d_{A\cup U\cup V}(x, y) + d_{B\cup U\cup W}(y, z) + d_{C\cup V\cup W}(z, x) = d_H(x, y) + d_I(y, z) + d_J(z, x) = 3. \quad \square$$

Suppose $\Omega \subset \Omega'$. Then the constraints of IP corresponding to Ω are a subset of the constraints of IP corresponding to Ω'. However, we cannot infer that that any ASWF for Ω' will specify an ASWF for Ω. For example, (x, y) may be a trivial pair in Ω but not in Ω'. In this case $d_S(x, y) = 1$ is an additional constraint in the IP for Ω but not in the IP for Ω'.

We now show how the integer programming formulation can be used to derive Arrow's Impossibility Theorem.

THEOREM 2 (ARROW'S IMPOSSIBILITY THEOREM). *When* $\Omega = \Sigma$, *the* 0–1 *solutions to the IP correspond to dictatorial rules.*

PROOF. When $\Omega = \Sigma$, we know from constraints (5)–(6) and the existence of all possible triples that $d_S(x, y) = d_S(y, z) = d_S(z, u)$ for all alternatives x, y, z, u. We will thus write d_S in place of $d_S(x, y)$ in the rest of the proof.

First, we show that $d_S = 1 \Rightarrow d_T = 1$ for all $T \supset S$. Suppose not. Let T be a set containing S with $d_T = 0$. Constraint (2) implies $d_{T^c} = 1$. Choose $A = T \setminus S$, $U = T^c$, and $V = S$ in (3). Then, $d_{A\cup U\cup V} = d_E = 1$, $d_{B\cup U\cup W} = d_{T^c} = 1$, and $d_{C\cup V\cup W} = d_S = 1$, which contradicts (3).

The same argument implies that $d_T = 0 \Rightarrow d_S = 0$ whenever $S \subset T$. Note also that if $d_S = d_T = 1$, then $S \cap T \neq \varnothing$, otherwise the assignment $A = (S \cup T)^c$, $U = S$, $V = T$ will violate the cycle elimination constraint. Furthermore, $d_{S\cap T} = 1$, otherwise the assignment $A = (S \cup T)^c$, $U = T \setminus S$, $V = S \setminus T$, $W = S \cap T$ will violate the cycle elimination constraint. Hence, there exists a minimal set S^* with $d_{S^*} = 1$ such that all T with $d_T = 1$ contains S^*. We show that $|S^*| = 1$. If not, there will be $j \in S$ with $d_j = 0$, which by (2) implies $d_{E\setminus\{j\}} = 1$. Because $d_{S^*} = 1$ and $d_{E\setminus\{j\}} = 1$, $d_{E\setminus\{j\}\cap S^*} \equiv d_{S^*\setminus\{j\}} = 1$, contradicting the minimality of S^*. $\square$

REMARK. We can interpret the above Impossibility Theorem as one that characterizes the 0–1 solutions to the IP: There exists j^* such that $d_S(x, y) = 1$ for every S containing j^*. This interpretation allows us to obtain Impossibility Theorems for other classes of problems in social choice theory; see §5 for a discussion.

For subsequent applications we introduce the *born-loser rule*. For each j, we define the born-loser rule with respect to j (denoted by B_j) in the following way:

- $d_E^{B_j}(x, y) = 1$ for every $x, y \in \mathcal{A}^2$.
- $d_\varnothing^{B_j}(x, y) = 0$ for every $x, y \in \mathcal{A}^2$.
- For every nontrivial pair (x, y), and for any $S \neq \varnothing, E$, $d_S^{B_j}(x, y) = 0$ if $S \ni j$, $d_S^{B_j}(x, y) = 1$ otherwise.

The born-loser rule is identical in all respects but one to the anti-dictatorship rule of Wilson (1972). The born-loser rule is required to satisfy unanimity, anti-dictatorship is not.

THEOREM 3. *For any j and $n > 2$, the born-loser rule B_j is a nondictatorial n-person ASWF if and only if for all x, y, z, there do not exist $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ in Ω with*

$$x\,\mathbf{p}_1\,z\,\mathbf{p}_1\,y, \quad x\,\mathbf{p}_2\,y\,\mathbf{p}_2\,z, \quad z\,\mathbf{p}_3\,x\,\mathbf{p}_3\,y.$$

PROOF. It is clear that by definition, d^{B_j} satisfies (1) and (2). To see that it satisfies (3), observe that in every partition of the agents, one of the sets obtained must contain j. Say $j \in A \cup U \cup V$. If $d_{A\cup U\cup V}^{B_j}(x, y) = 0$, then (3) is clearly valid. So we may assume that $d_{A\cup U\cup V}^{B_j}(x, y) = 1$. This happens only when $A \cup U \cup V = E$ (or if (x, y) is trivial, which in turn implies that all the other sets are empty). We may assume $U, V \neq \varnothing$ and $j \in A$, otherwise (3) is clearly valid. But according to condition (*), this implies existence of $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ in Ω with

$$x\, \mathbf{p}_1\, z\, \mathbf{p}_1\, y, \quad x\, \mathbf{p}_2\, y\, \mathbf{p}_2\, z, \quad z\, \mathbf{p}_3\, x\, \mathbf{p}_3\, y,$$

which is a contradiction.

So, d^{B_j} satisfies (1)–(3) and hence corresponds to an ASWF. When $n > 2$, B_j is clearly nondictatorial. □

2.1. General domains. The IP characterization obtained above can be generalized to the case in which the domain of preferences for each voter is different. In general, let D be the domain of profiles over alternatives. In this case, for any set S, the d_S variables may not be well defined for a pair of alternatives x, y, if there is no profile in which all agents in S (resp. S^c) rank x over y (resp. y over x). Thus, $d_S(x, y)$ is well defined only if such profiles exist. Note that $d_S(x, y)$ is well defined if and only if $d_{S^c}(y, x)$ is well defined. With this proviso, inequalities (1) and (2) remain valid. We only need to modify (3) to the following:

Let A, B, C, U, V, and W be (possibly empty) *disjoint* sets of agents whose union includes all agents. For each such partition of the agents, and any triple x, y, z,

$$d_{A\cup U\cup V}(x, y) + d_{B\cup U\cup W}(y, z) + d_{C\cup V\cup W}(z, x) \leq 2, \tag{9}$$

where the sets satisfy the following conditions:

$$A \neq \varnothing \text{ only if there exists } \mathbf{p}_i, i \in A, \text{ with } x\, \mathbf{p}_i\, z\, \mathbf{p}_i\, y,$$
$$B \neq \varnothing \text{ only if there exists } \mathbf{p}_i, i \in B, \text{ with } y\, \mathbf{p}_i\, x\, \mathbf{p}_i\, z,$$
$$C \neq \varnothing \text{ only if there exists } \mathbf{p}_i, i \in C, \text{ with } z\, \mathbf{p}_i\, y\, \mathbf{p}_i\, x,$$
$$U \neq \varnothing \text{ only if there exists } \mathbf{p}_i, i \in U, \text{ with } x\, \mathbf{p}_i\, y\, \mathbf{p}_i\, z,$$
$$V \neq \varnothing \text{ only if there exists } \mathbf{p}_i, i \in V, \text{ with } z\, \mathbf{p}_i\, x\, \mathbf{p}_i\, y,$$
$$W \neq \varnothing \text{ only if there exists } \mathbf{p}_i, i \in W, \text{ with } y\, \mathbf{p}_i\, z\, \mathbf{p}_i\, x,$$
$$\text{and } (\mathbf{p}_1, \ldots, \mathbf{p}_n) \in D.$$

The following theorem is immediate from our discussion; we omit the proof.

THEOREM 4. *Every feasible integer solution to* (1), (2), *and* (9) *corresponds to an ASWF on domain D and vice versa.*

3. Restricted ASWFs. Additional conditions that are sometimes imposed on an ASWF are anonymity, neutrality, and monotonicity. In this section, we consider ASWFs subject to various combinations of these three conditions; our goal, once again, is to characterize the domains admitting such ASWFs.

3.1. Definitions

- An ASWF is called *anonymous* if its ranking over pairs of alternatives remains unchanged when the labels of the agents are permuted. Hence, $d_S(x, y) = d_T(x, y)$ for all $(x, y) \in \mathcal{A}^2$ whenever $|S| = |T|$. Observe that a dictatorial rule is not anonymous.
- An ASWF is called *neutral* if its ranking over any pair of alternatives depends only on the pattern of agents' preferences over that pair, not on the alternatives' labels. Neutrality implies that $d_S(x, y) = d_S(a, b)$ for any $(x, y), (a, b) \in \mathcal{A}^2$. Thus, the value of $d_S(\cdot, \cdot)$ is determined by S alone.
- An ASWF is called *monotonic* if when one switches from the profile $\mathbf{P}$ to $\mathbf{Q}$ by raising the ranking of $x \in A$ for at least one agent, then $f(\mathbf{Q})$ will not rank x lower than it is in $f(\mathbf{P})$. Equivalently, $d_S(x, y) \leq d_T(x, y)$ whenever $S \subset T$.

3.2. Anonymous and neutral ASWFs. When anonymity and neutrality are combined, $d_S(\cdot,\cdot)$ is determined by $|S|$ alone. In such a case, we write d_S as d_r, where $r = |S|$. If n is even, it is not possible for an anonymous ASWF to be neutral because Equation (2) cannot be satisfied for $|S| = n/2$; so we assume that n is odd in the rest of this subsection.

The majority rule is both anonymous and neutral, but is not the only such rule. For three agents, the three-person minority rule is anonymous and neutral. (In the three-person minority rule, only singleton sets and the entire set of agents are decisive.)

Sen (1966) characterizes those domains for which the majority rule is an ASWF. Maskin (1995) characterizes those domains that admit anonymous and neutral ASWFs. Here we go one step further and characterize those domains that admit a nondictatorial, neutral ASWF. The proof uses the integer programming formulation introduced earlier. As stepping stones we need the results of Sen and Maskin, which we also (re)derive using the IP for the sake of completeness.

Recall that Ω admits a *Condorcet* triple if there are x, y and $z \in \mathcal{A}$ and $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3 \in \Omega$ such that $x\,\mathbf{p}_1\,y\,\mathbf{p}_1\,z$, $y\,\mathbf{p}_2\,z\,\mathbf{p}_2\,x$, and $z\,\mathbf{p}_3\,x\,\mathbf{p}_3\,y$. The following theorem is essentially due to Sen (1966).

THEOREM 5. *For an odd number of agents, the majority rule is an ASWF on Ω if and only if Ω does not contain a Condorcet triple.*

PROOF. First, suppose that the majority rule is an ASWF on Ω. For a contradiction, assume that $x, y, z \in \mathcal{A}$ form a Condorcet triple. Let n, the number of agents, be $3r+k$ for some integers $r \geq 1$ and $k = 0, 1$, or 2. If $k = 0$, partition the agents into three sets of size r called U, V, and W. Every agent in U ranks x above y above z. Every agent in V ranks z above x above y.

Every agent in W ranks y above z above x. Because n is odd and $2r > n/2$, it follows that on this profile that the majority rule produces a cycle.

If $k = 1$, choose U, V, and W as above but $|U| = |V| = r$ and $|W| = r+1$.Once again $2r+1 > 2r > n/2$, so the majority rule cycles again. If $k = 2$ repeat the argument with $|U| = r$ and $|V| = |W| = r+1$.

Now suppose that Ω has no Condorcet triple. To show that the majority rule is an ASWF we must show that inequality (3) is satisfied. To obtain a contradiction suppose not. Fix a triple $x, y, z \in \mathcal{A}$ for which (3) is violated. Because Ω has no Condorcet triple, at least one of A, B, or C is empty and at least one of U, V, and W is empty. Without loss of generality suppose that $A, W = \varnothing$. Because (3) is violated we have $d_{U\cup V}(x, y) = d_{B\cup U}(y, z) = d_{C\cup V}(z, x) = 1$. The majority rule implies that $|B|+|U| > n/2$ and $|C|+|V| > n/2$. Adding these two inequalities yields $n = |B|+|U|+|C|+|V| > n$, a contradiction. □

The next result we derive using the IP formulation is essentially due to Maskin (1995).

THEOREM 6. *Suppose there are at least* 3 *agents. If Ω admits an anonymous, neutral ASWF, then Ω has no Condorcet triples.*

PROOF. Suppose Ω admits an anonymous, neutral ASWF f, and suppose $x, y, z \in \mathcal{A}$ is a collection that forms a Condorcet triple. Consider the d variables associated with the ASWF f. Thus, d satisfies (1)–(3). Let n denote the number of agents.

Inequality (3) implies $1 \leq d_a(x, y)+d_b(y, z)+d_c(z, x) \leq 2$, whenever $a, b, c > 0$, $a+b+c = n$. By neutrality, $d_S(x, y) = d_S(a, b)$ for all $(x, y), (a, b)$. So we omit the alternatives and represent the variables as $d_{|S|}$.

By (2), $d_1 \neq d_{n-1}$. Furthermore, $1 \leq d_1+d_1+d_{n-2} \leq 2 \Rightarrow d_{n-2} \neq d_1$, i.e., $d_{n-1} = d_{n-2}$. By (2), we must have $d_2 = d_1$. Similarly, $1 \leq d_1+d_2+d_{n-3} \leq 2 \Rightarrow d_{n-3} = d_{n-2} = d_{n-1}$. Repeating the above argument, we obtain the series of equalities

$$d_1 = d_2 = d_3 = \cdots = d_{\lfloor n/2 \rfloor},$$
$$d_{n-1} = d_{n-2} = \cdots = d_{\lceil n/2 \rceil}.$$

The number of agents, n, must be odd, otherwise no anonymous ASWF is neutral. If n is odd, however, $d_1 + d_{\lfloor n/2 \rfloor} + d_{\lfloor n/2 \rfloor} = 0$ or 3, a contradiction. □

An immediate consequence of Theorems 5 and 6 is the following.

COROLLARY 1. *Let the number of agents be odd. Then the following three statements are equivalent*:

1. *The domain* Ω *admits an anonymous and neutral ASWF.*
2. *The majority rule is an ASWF on* Ω.
3. *The domain* Ω *does not contain any Condorcet triples.*

3.3. Neutral ASWFs. We now characterize domains that admit *neutral* ASWFs. We show that checking whether a domain Ω admits a neutral, nondictatorial ASWF reduces to checking whether the majority rule *or* the born-loser rule is an ASWF on that domain. For the majority rule to be defined with an even number of voters, we must add a tie-breaking rule. This is done by adding a dummy voter endowed with an arbitrary preference ordering from Ω.

THEOREM 7. *For* $n \geq 3$, *a domain* Ω *admits a neutral, nondictatorial ASWF if and only if the majority rule or the born-loser rule is an ASWF on* Ω.

PROOF. If either the majority rule or the born-loser rule are ASWFs on Ω, Ω clearly admits a neutral, nondictatorial ASWF. Suppose then Ω admits a neutral, nondictatorial ASWF, but neither the majority rule nor the born-loser rule is an ASWF on Ω. Because the majority rule is not an ASWF, Ω admits a Condorcet triple $\{a, b, c\}$. Because the born-loser rule is not an ASWF on Ω, by Corollary 1 there exist $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ in Ω and $x, y, z \in \mathcal{A}$ with

$$x\,\mathbf{p}_1\,z\,\mathbf{p}_1\,y, \quad x\,\mathbf{p}_2\,y\,\mathbf{p}_2\,z, \quad z\,\mathbf{p}_3\,x\,\mathbf{p}_3\,y.$$

We will need the existence of these orderings to construct a partition of the agents that satisfies the cycle elimination constraints. The proof will mimic the proof of Arrow's Impossibility Theorem (Theorem 2) given earlier.

Neutrality implies that $d_S(x, y) = d_S(y, z) = d_S(z, u)$ for all alternatives x, y, z, u. We will thus write d_S in place of $d_S(x, y)$ in the rest of the proof.

First, $d_S = 1 \Rightarrow d_T = 1$ for all $T \supset S$. Suppose not. Let T be the set containing S with $d_T = 0$. Constraint (2) implies $d_{T^c} = 1$. Choose $A = T \setminus S$, $U = T^c$, and $V = S$ in (3). We can do this because of $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$. Then, $d_{A \cup U \cup V} = d_E = 1$, $d_{B \cup U \cup W} = d_{T^c} = 1$, and $d_{C \cup V \cup W} = d_S = 1$, which contradicts (3).

The same argument implies that $d_T = 0 \Rightarrow d_S = 0$ whenever $S \subset T$. Note also that if $d_S = d_T = 1$, then $S \cap T \neq \varnothing$, otherwise the assignment $A = (S \cup T)^c, U = S, V = T$ will violate the cycle elimination constraint.

Next we show that $d_{S \cap T} = 1$. Suppose not. Consider the assignment $U = E \setminus S$, $V = S \setminus T$, and $W = S \cap T$. We can choose such a partition because $\{a, b, c\}$ form a Condorcet triple. For this specification, $d_{A \cup U \cup V} = d_{E \setminus \{S \cap T\}} = 1$. Because $T \subset B \cup U \cup W$, $d_{B \cup U \cup W} = 1$, and $d_{C \cup V \cup W} = d_S = 1$, which contradicts (3).

Hence, there exists a minimal set S^* such that $d_{S^*} = 1$ and every T with $d_T = 1$ contains S^*. We show that $|S^*| = 1$. If not, there will be $j \in S$ with $d_j = 0$, and hence $d_{E \setminus \{j\} \cap S^*} = 1$, contradicting the minimality of S^*. □

A simple consequence of this result is the following theorem due to Kalai and Muller (1977). The proof is new.

THEOREM 8. *A nondictatorial solution to* (1), (2), *and* (5)–(8) *exists for the case* $n = 2$ *agents if and only if a nondictatorial solution to* (1)–(3) *exists for any n.*

PROOF. Given a 2-person nondictatorial ASWF, we can build an ASWF for the n-person case by focusing only on the preferences submitted by the first two voters and rank the alternatives using the 2-person ASWF. This is clearly a nondictatorial ASWF for the n-person case. Hence, we only need to give a proof of the converse.

Let d^* be a nondictatorial solution to (1)–(3). Suppose d does not imply a neutral ASWF. Then, there is a set of agents S such that $d^*_S(x, y)$ is nonzero for some but not all $(x, y) \in \mathcal{A}^2$. Hence, $d_1 = d^*_S$, $d_2 = d^*_{S^c}$ would be a nondictatorial solution to (1), (2), and (5)–(8).

Suppose then d implies a neutral ASWF. By the previous theorem we can choose d to be either the majority rule or the born-loser rule. In the first case, we can build a 2-person ASWF by using a dummy voter with a fixed ordering from Ω and using the (3-person) majority rule. In the second case, we can build a 2-person ASWF by adding a dummy born-loser. □

The following refinement to Maskin's (1995) result (Theorem 6) also follows directly from Theorem 7. It says that for many classes of domains, the majority rule is essentially the only anonymous and neutral ASWF.

THEOREM 9. *Let the number of agents be odd. Suppose Ω does not contain any Condorcet triples, and suppose there exist $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ in Ω and $x, y, z \in \mathcal{A}$ with*

$$x\, \mathbf{p}_1\, z\, \mathbf{p}_1\, y, \quad x\, \mathbf{p}_2\, y\, \mathbf{p}_2\, z, \quad z\, \mathbf{p}_3\, x\, \mathbf{p}_3\, y.$$

Then, the majority rule is the only *anonymous, neutral ASWF on Ω.*

PROOF (SKETCH). From the proof of Theorem 7, we know that if d_S corresponds to a neutral ASWF, and if there exist $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ in Ω and $x, y, z \in \mathcal{A}$ with $x\, \mathbf{p}_1\, z\, \mathbf{p}_1\, y$, $x\, \mathbf{p}_2\, y\, \mathbf{p}_2\, z$, $z\, \mathbf{p}_3\, x\, \mathbf{p}_3\, y$, then d_S is monotonic, i.e., $d_S \leq d_T$ if $S \subset T$. By May's Theorem, it has to be the majority rule because the majority rule is the only ASWF that is anonymous, neutral, and monotonic. □

4. Single-peaked domains. The domain Ω is single peaked with respect to a linear ordering $\mathbf{q}$ over $\mathcal{A}$ if $\Omega \subseteq \Omega(\mathbf{q}) \equiv \{\mathbf{p} \in \Sigma :$ for every triple (x, y, z) if $x\, \mathbf{q}\, y\, \mathbf{q}\, z$ then it is *not* the case that $x\, \mathbf{p}\, y$ and $z\, \mathbf{p}\, y\}$. The class of single-peaked preferences has received a great deal of attention in the literature. Here we show how the IP can be used to characterize the class of ASWFs on single-peaked domains. In §4.1, we prove that the constraints (1)–(3), along with the nonnegative constraints on the d variables, are *sufficient* to characterize the convex hull of 0–1 solutions. We next use this proof technique to generalize Moulin's (1984) result on Generalized majority rules.

4.1. Integrality of the IP on single-peaked domains.

THEOREM 10. *When Ω is single peaked the set of nonnegative solutions satisfying* (1)–(3) *is an integral polytope. All ASWFs are extreme point solutions of this polytope.*

REMARK. If the domain contains all possible single-peaked orderings (with respect to a given linear order) so that constraints (7) and (8) are redundant, the integrality result follows easily because the constraint matrix is now totally unimodular. Here, we show that this result holds even if the domain consists of some (but not all) single-peaked orderings with respect to a given linear order.

PROOF. It suffices to prove that every fractional solution satisfying (1)–(3) can be written as a convex combination of 0–1 solutions satisfying the same set of constraints. Let $\mathbf{q}$ be the linear ordering with respect to which Ω is single peaked.

Rounding scheme. Let $d_S(\cdot)$ be a (possibly) fractional solution to the linear programming relaxation of (1)–(3). We round the solution d to the 0–1 solution d' in the following way:

- Generate a random number Z uniformly between 0 and 1.
- For $a, b \in \mathscr{A}$ with $a\,\mathbf{q}\,b$ and $S \subset E$, then
  — $d'_S(a, b) = 1$, if $d_S(a, b) > Z$, 0 otherwise;
  — $d'_S(b, a) = 1$, if $d_S(b, a) \geq 1 - Z$, 0 otherwise.

The integral solution obtained is feasible. The 0–1 solution d'_S generated in the manner above clearly satisfies constraint (1). To verify that it satisfies constraint (2), consider a set $T \subseteq E$, an arbitrary pair of alternatives a, b, and suppose without loss of generality $a\,\mathbf{q}\,b$. From the linear programming relaxation, we know that either $d_T(a, b) > Z$ or $d_T(b, a) \geq 1 - Z$ (because the two variables add up to 1), but not both. Thus, exactly one of $d'_T(a, b)$ or $d'_T(b, a)$ is set to 1.

Next, we show that all the constraints in (3) are satisfied by the solution $d'_S(\cdot)$. Consider three alternatives a, b, c, and constraint (3) (with a, b, c replacing the role of x, y, z) can be rewritten as $d_{A\cup U\cup V}(a, b) + d_{B\cup U\cup W}(b, c) + d_{C\cup V\cup W}(c, a) \leq 2$.

Suppose $a\,\mathbf{q}\,b\,\mathbf{q}\,c$. Then, in constraint (3), by the single-peakedness property, we must have $A = V = \varnothing$. In this case, the constraint reduces to $d_U(a, b) + d_{B\cup U\cup W}(b, c) + d_{C\cup W}(c, a) \leq 2$.

We need to show that $d'_U(a, b) + d'_{B\cup U\cup W}(b, c) + d'_{C\cup W}(c, a) \leq 2$. By choosing the sets in constraint (3) in a different way, with

$$U' \leftarrow U, \quad B' \leftarrow B, \quad W' \leftarrow W \cup C, \quad C' \leftarrow \varnothing,$$

we deduce that $d_{U'}(a, b) + d_{B'\cup U'\cup W'}(b, c) + d_{C'\cup W'}(c, a) \leq 2$, which is equivalent to $d_U(a, b) + 1 + d_{C\cup W}(c, a) \leq 2$.

Hence, we must have $d_U(a, b) + d_{C\cup W}(c, a) \leq 1$. Note that because $a\,\mathbf{q}\,b$ and $b\,\mathbf{q}\,c$, our rounding scheme ensures that $d'_U(a, b) + d'_{C\cup W}(c, a) \leq 1$. Therefore,

$$d'_U(a, b) + d'_{B\cup U\cup W}(b, c) + d'_{C\cup W}(c, a) \leq 2.$$

To finish the proof, we need to show that constraint (3) holds for different orderings of a, b, and c under $\mathbf{q}$; the above argument can be easily extended to handle all these cases to show that constraint (3) is valid. We omit the details here.

All extreme point solutions are integral. Suppose now that the fractional solution d_S is a fractional extreme point in the polytope defined by constraints (1)–(3). By standard polyhedral theory, there exists a cost function $(c_S(a, b))$ such that d_S is the unique minimum solution to the problem:

$$(P) \quad \min \quad \sum_{S,\,a,\,b} c_S(a,\ b) x_S(a,\ b)$$

$$\text{subject to} \quad x_S(a, b) \text{ satisfies constraints (1)–(3);} \quad x_S(a,\ b) \in [0, 1].$$

The rounding scheme we have just described converts a fractional solution to a 0–1 solution satisfying

$$E(d'_S(a, b)) = P(Z < d_S(a, b)) = d_S(a, b) \quad \text{if } a\,\mathbf{q}\,b,$$

and

$$E(d'_S(a, b)) = P(Z \geq 1 - d_S(a, b)) = d_S(a, b) \quad \text{if } b\,\mathbf{q}\,a.$$

Hence, $E(d'_S(a, b)) = d_S(a, b)$ for all S, a, and b. Thus,

$$E\left(\sum_{S,a,b} c_S(a, b)d'_S(a, b)\right) = \sum_{S,a,b} c_S(a, b)d_S(a, b),$$

and hence all the 0–1 solutions obtained by the rounding scheme must also be a minimum solution to problem (P). This contradicts the fact that d_S is the unique minimum solution. □

While this paper was under revision, a paper by Ehlers and Storcken (2001) came to our attention. They also derive a charcterization of ASWFs for single-peaked preferences, but with a continuum of alternatives and "continuous" preferences.

The argument above shows the set of ASWFs on single-peaked domains (wrt $\mathbf{q}$) has a property similar to the generalized median property of the stable marriage problem (see Teo and Sethuraman 1998).

THEOREM 11. *Let $f_1, f_2, \ldots, f_N$ be distinct ASWFs for the single-peaked domain Ω (with respect to $\mathbf{q}$). Define a function F_k: $\Omega^n \to \Sigma$ with the property*:

The set S under F_k is decisive for x over y if $x\,\mathbf{q}\,y$, and S is decisive for x over y for at least $k+1$ of the ASWF f_i's; or $y\,\mathbf{q}\,x$, and S is decisive for x over y for at least $N-k$ of the ASWF f_i's.

Then F_k is also an ASWF.

One consequence of Theorem 11 is that when Ω is single peaked, it is Arrovian, because the dictatorial ASWF can be used to construct nondictatorial ASWFs in the above manner. For instance, consider the case $n = 2$. Let f_1 and f_2 be the dictatorial rule associated with agents 1 and 2, respectively. The function F_1 constructed above reduces to the following ASWF:

If $x\,\mathbf{q}\,y$, the social welfare function ranks x above y if and only if both agents prefer x over y.

If $y\,\mathbf{q}\,x$, the social welfare function ranks y above x if and only if none of the agents prefer x above y.

4.2. Generalized majority rule. Moulin (1984) has introduced an extension of the majority rule called the generalized majority rule (GMR). A GMR M for n agents of the following form:

- Add $n-1$ dummy agents, each with a fixed preference drawn from Σ.
- x is ranked above y under M if and only if the majority (of real and dummy agents) prefer x to y.

Each instance of a GMR can be described algebraically as follows. Fix a profile $\mathbf{R} \in \Sigma^{n-1}$ and let $R(x, y)$ be the number of orderings in $\mathbf{R}$ where x is ranked above y. Given any profile $\mathbf{P} \in \Omega^n$, GMR ranks x above y if the number of agents who rank x above y under $\mathbf{P}$ is at least $n - R(x, y)$.

Moulin (1984) considered the domain consisting of *all* single-peaked orderings with respect to a given linear order $\mathbf{q}$, and proved that every anonymous, monotonic ASWF on such a domain must be a GMR. Here, we generalize this result to domains consisting of some (not necessarily all) single-peaked orderings with respect to the given linear order $\mathbf{q}$. The proof relies on the geometric construction used in the integrality proof of §4.1.

THEOREM 12. *An ASWF that is anonymous and monotonic on a single-peaked domain Ω must be a GMR.*

PROOF. Let d_S be a solution to (1)–(3), corresponding to an anonymous and monotonic ASWF on the domain Ω. Let $\mathbf{q}$ be the underlying order of alternatives. For each $(x, y) \in \mathcal{A}^2$, by anonymity, $d_S(x, y)$ depends only on the cardinality of S. Monotonicity implies $d_S(x, y) \le d_T(x, y)$ if $S \subseteq T$. Thus, $d_S(x, y) = 1$ if and only if $|S| \ge e(x, y)$ for some number $e(x, y)$. To complete the proof we need to determine a profile $\mathbf{R} \in \Sigma^{n-1}$ such that $n - R(x, y) = e(x, y)\ \forall (x, y) \in \mathcal{A}^2$.

Because $d_S(x, y) + d_{S^c}(y, x) = 1$, we have $e(x, y) + e(y, x) = n + 1$ for all (x, y) and (y, x). Note that $e(x, y) \ge 1$ and $e(x, y) \le n$.

We use the geometric construction used in the earlier proof to construct the profile $\mathbf{R} \in \Sigma^{n-1}$

- to each (x, y) such that $x\,\mathbf{q}\,y$, associate the interval $[0, e(x, y)]$ and label it $l(x, y)$; and
- to each (x, y) such that $y\,\mathbf{q}\,x$, associate the interval $[n+1-e(x, y), n+1]$ and label it $l(x, y)$.

We construct preferences in $\mathbf{R}$ in the following way:

- For each $k = 1, 2, \ldots, n-1$, if $l(x, y)$ covers the point $k + 0.5$, then the kth dummy voter ranks y over x. Otherwise, the dummy voter ranks x over y.

Thus, each dummy voter k ranks x above y if and only if the interval $l(y, x)$ *does not* cover the point $k + 0.5$.

Because the intervals $l(x, y)$ and $l(y, x)$ are disjoint and cover $[0, n+1]$ the procedure is well defined. If $R(x, y)$ is the number of dummy voters who rank x above y in this construction it is easy to see that $n - R(x, y) = e(x, y)$, which is what we need. It remains then to to show that the profile constructed is in $\Omega(\mathbf{q})^{n-1}$.

CLAIM 3. The procedure returns a linear ordering of the alternatives.

PROOF. Suppose otherwise and consider three alternatives x, y, z where the procedure (for some dummy voter) ranks x above y, y above z, and z above x. Hence, the intervals $l(x, y)$, $l(y, z)$, and $l(z, x)$ do not cover the point $k + 0.5$. From symmetry, it suffices to consider the following two cases:

- *Case* 1. Suppose $x\,\mathbf{q}\,y\,\mathbf{q}\,z$. By the geometric arrangement of the intervals, we have $e(x, z) > e(y, z)$, $e(x, z) > e(x, y)$. Hence, there exists S such that $d_S(x, y) = 1$, $d_S(y, z) = 1$, and $d_S(x, z) = 0$, for S with suitable cardinality. This is a contradiction if there exists $\mathbf{p} \in \Omega$ with $x\,\mathbf{p}\,y\,\mathbf{p}\,z$. Suppose not. Note that the ordering of x and z above y is ruled out by single-peakness property of Ω. Hence, (y, x) is a trivial pair. This is a contradiction because $e(x, y) < e(x, z) \le n$.
- *Case* 2. Suppose $y\,\mathbf{q}\,x\,\mathbf{q}\,z$. Now, by the geometric arrangement of the intervals, $e(y, z) < e(y, x)$, $e(y, z) < e(x, z)$. Hence, there exists S such that $d_S(y, z) = 1$, $d_S(y, x) = 0$, and $d_S(x, z) = 0$, for S with suitable cardinality. This is a contradiction if there exists $\mathbf{p} \in \Omega$ with $z\,\mathbf{p}\,x\,\mathbf{p}\,y$. Suppose not. Note that the ordering of y and z above x is ruled out by single-peakness property of Ω. Hence, (x, z) is a trivial pair. This is a contradiction because $1 \le e(y, z) < e(x, z)$.

We conclude that the ordering constructed is a linear order. □

REMARK. Moulin (1984) showed that for the domain consisting of *all* single-peaked orderings with respect to the linear order $\mathbf{q}$, the profile, $\mathbf{R}$, for the dummy voters can be chosen from Ω^{n-1} rather than Σ^{n-1}. The $\mathbf{R}$ constructed in the above proof can be shown to belong to Ω^{n-1} in that special case.

5. Muller-Satterthwaite theorem. A *social choice function* maps profiles of preferences into a single alternative. It is natural to ask if the integer programming approach described above can be used to obtain results about social choice functions. Up to a point, yes. The difficulty is that knowing what alternative a social choice function will pick from a set of size two, does not, in general, allow one to infer what it will choose when the set of alternatives is extended by one. However, given additional assumptions one can surmount this difficulty. We illustrate with one example.

The analog of Arrow's Impossibility Theorem for social choice functions is the Muller-Satterthwaite theorem (1977). The counterpart of unanimity and the independence of irrelevant alternatives condition for social choice functions are called *Pareto optimality* and *monotonicity*. To define them, denote the preference ordering of agent i in profile $\mathbf{P}$ by $\mathbf{p}_i$.

1. **Pareto optimality.** Let $\mathbf{P} \in \Omega^n$ such that $x\,\mathbf{p}\,y$ for all $\mathbf{p} \in \mathbf{P}$. Then $f(\mathbf{P}) \neq y$.
2. **Monotonicity.** For all $x \in \mathcal{A}$, $\mathbf{P}, \mathbf{Q} \in \Omega^n$ if $x = f(\mathbf{P})$ and $\{y\colon x\,\mathbf{p}_i\,y\} \subseteq \{y\colon x\,\mathbf{q}_i\,y\}$ $\forall i$ then $x = f(\mathbf{Q})$.

We call a social choice function that satisfies Pareto optimality and monotonicity an Arrovian social choice function (ASCF).

THEOREM 13 (MULLER-SATTERTHWAITE). *When $\Omega = \Sigma$, all ASCFs are dictatorial.* (*The more well-known result about strategy proof social choice functions is due to Gibbard* (1973) *and Satterthwaite* (1975); *it is a consequence of Muller and Satterthwaite* 1977.)

PROOF. For each subset S of agents and ordered pair of alternatives (x, y), denote by $[S, x, y]$ the set of all profiles where agents in S rank x first and y second and agents in S^c rank y first and x second. By the hypothesis on Ω this collection is well defined.

For any profile $\mathbf{P} \in [S, x, y]$, it follows by Pareto optimality that $f(\mathbf{P}) \in \{x, y\}$. By monotonicity, if $f(\mathbf{P}) = x$ for one such profile $\mathbf{P}$, then $f(\mathbf{P}) = x$ for all $\mathbf{P} \in [S, x, y]$.

Suppose then for all $\mathbf{P} \in [S, x, y]$, we have $f(\mathbf{P}) \neq y$. Let $\mathbf{Q}$ be any profile where all agents in S rank x above y and all agents in S^c rank y above x. We show next that $f(\mathbf{Q}) \neq y$ too.

Suppose not. That is, $f(\mathbf{Q}) = y$. Let $\mathbf{Q}'$ be a profile obtained by moving x and y to the top in every agents ordering but preserving their relative position within each ordering. So, if x was above y in the ordering under $\mathbf{Q}$, it remains so under $\mathbf{Q}'$. Similarly, if y was above x. By monotonicity $f(\mathbf{Q}') = y$. However, monotonicity with respect to $\mathbf{Q}'$ and $\mathbf{P} \in [S, x, y]$ implies that $f(\mathbf{P}) = y$, a contradiction.

Hence, if there is one profile in which all agents in S rank x above y all agents in S^c rank y above x, and y is not selected, then all profiles with such a property will not select y. This observation allows us to describe ASCFs using the following variables.

For each $(x, y) \in \mathcal{A}^2$, define a 0–1 variable as follows:

- $g_S(x, y) = 1$ if when all agents in S rank x above y and all agents in S^c rank y above x, then y is never selected.
- $g_S(x, y) = 0$ otherwise.

If E is the set of all candidates, we set $g_E(x, y) = 1$ for all $(x, y) \in \mathcal{A}^2$. This ensures Pareto optimality.

Consider a $\mathbf{P} \in \Omega^n$, $(x, y) \in \mathcal{A}^2$, and subset S of agents such that all agents in S prefer x to y and all agents in S^c prefer y to x. Then, $g_S(x, y) = 0$ implies that $g_{S^c}(y, x) = 1$ to ensure a selection. Hence, for all S and $(x, y) \in \mathcal{A}^2$ we have $g_S(x, y) + g_{S^c}(y, x) = 1$

We show that the variables g_S satisfy the cycle elimination constraints. If not there exists a triple $\{x, y, z\}$, and sets A, B, C, U, V, W such that the cycle elimination constraint is violated. Consider the profile $\mathbf{P}$ where each voter ranks the triple $\{x, y, z\}$ above the rest, and with the ordering of x, y, z depending on whether the voter is in A, B, C, U, V, or W. Because $g_{A\cup U\cup V}(x, y) = 1$, $g_{B\cup U\cup W} = 1$, and $g_{C\cup V\cup W} = 1$, none of the alternatives x, y, z is selected for the profile $\mathbf{P}$. This violates Pareto optimality, a contradiction. Hence, g_S satisfies constraints (1)–(3). Because $\Omega = \Sigma$, by Arrow's Impossibility Theorem, g_S corresponds to a dictatorial solution. □

The above approach can also be used to derive other classes of impossibility theorems. For instance, a recent paper by Dutta et al. (2001) considers the issue when the alternatives correspond to the set of candidates for election. Candidates can withdraw themselves and in so doing affect the outcome of an election. They ask what election rules would be immune to such manipulations. Under very general conditions, they proved that only dictatorial voting rules are possible. For a complete discussion, and a derivation using the IP approach, we refer the reader to the technical report of the authors (Sethuraman et al. 2001).

6. Complexity. In this section, we turn our attention to the complexity of deciding whether or not a given domain Ω is Arrovian. The answer to this question depends on how the domain Ω is encoded; these issues are further discussed in §6.1. In §6.2, we present some preliminary results on the polyhedral structure of the IP when the number of alternatives is small.

6.1. Encoding Ω. A domain is called *decomposable* if and only if there is a nontrivial solution (not all 1's or all 0's) to the system of inequalities (1), (2), and (5)–(8) for the case $n = 2$. The main result of Kalai and Muller (1977, cf. Theorem 10) can be phrased as follows: *the domain Ω is nondictatorial if and only if it is decomposable*. This result allows one to formulate the problem of deciding whether Ω is Arrovian as an IP involving a number of variables and constraints that is polynomial in $|\mathcal{A}|$. (Interestingly, for all the preference domains considered in the literature that we are aware of, this IP reduces to a linear program. More details on this may be found in Sethuraman et al. (2001).) However, the set $\mathcal{A}$ is not the only input to the problem. The preference domain Ω is also an input. If Ω is specified by the set of permutations it contains, and if it has exponentially many permutations (say $O(2^{|\mathcal{A}|})$, the straight forward input model needs at least $O(2^{|\mathcal{A}|})$ bits. Recall that the number of decision variables for the IP for 2-person ASWFs is polynomial in $|\mathcal{A}|$. Furthermore, the time complexity of verifying the existence of triplets in Ω can trivially be performed in time $O(n^3 2^{|\mathcal{A}|})$. Hence, the decision version of the decomposability conditions can be solved in time polynomial in the size of the input.

Suppose, however, instead of listing the elements of Ω, we prescribe a polynomial time oracle to check membership in Ω. The complexity of deciding whether the domain is decomposable now depends on how we encode the membership oracle, and not on the number of elements in Ω. In this model, we exhibit an example to show that checking whether a triplet exists in Ω is already NP-hard.

Let G be a graph with vertex set V. Let Ω_G consist of all orderings of V that correspond to a Hamiltonian path in G. Given any triple $(u, v, w) \in V$, the problem of deciding if G admits a Hamiltonian path in which u precedes v precedes w is NP-complete. (If not, we can apply the algorithm for this problem thrice to decide if G admits a Hamiltonian cycle.) Hence, the problem of deciding whether there is a preference ordering $\mathbf{p}$ in Ω with $u\,\mathbf{p}\,v\,\mathbf{p}\,w$ is already NP-complete.

Thus, given an Ω specified by Hamiltonian paths, it is already NP-hard just to write down the set of inequalities specified by the decomposability conditions!

One way to bypass these difficulties is to restrict ourselves to collections of ordered triplets that are realized by some domain Ω. The input to the complexity question is thus a *feasible* set of ordered triplets ($O(n^3)$ size). We will focus on this input model for the rest of the paper.

6.2. Lifting. To understand the complexity of the problem further, we examine the polyhedral structure of the IP formulation when the number of alternatives is small.

We describe a sequential lifting method to derive new valid inequalities for the problem to strengthen the LP formulation, using the directed graph D^{Ω} defined below:

DEFINITION. With each *nontrivial* element of $\mathcal{A}^2$ we associate a vertex in D^{Ω}. Suppose there are $\mathbf{p}, \mathbf{q} \in \Omega$ and three alternatives x, y, and z such that $x\,\mathbf{p}\,y\,\mathbf{p}\,z$ and $y\,\mathbf{q}\,z\,\mathbf{q}\,x$. Then we insert an arc from (x, y) to (x, z), and another arc from (z, x) to (y, x). The graph obtained this way is denoted D^{Ω}. Such a graph was introduced by Muller (1982).

We say that the node u *dominates* the node v if there is a directed *path* in D^{Ω} from v to u (i.e., $d(u) \geq d(v)$).

Thus, difficulties with determining the existence of a feasible 0–1 solution are caused by inequalities (7)–(8). Any admissible ordering (by Ω) of three alternatives gives rise to them. However, some will be redundant. They are not redundant only when they are obtained from a triplet (x, y, z) with the property:

There exists $\mathbf{p}$ such that $x\,\mathbf{p}\,y\,\mathbf{p}\,z$ but no $\mathbf{q} \in \Omega$ such that $y\,\mathbf{q}\,z\,\mathbf{q}\,x$ or $z\,\mathbf{q}\,x\,\mathbf{q}\,y$.

Such a triplet is called an *isolated triplet*.

To obtain a better description of the polytope, we need to utilize the constraints imposed due to the presence of isolated triplets. To this end, we describe a sequential lifting method to obtain additional valid inequalities for the polytope.

Sequential lifting method.

- For each isolated triplet (x, y, z), we have the inequality

$$1 + d(x, z) \geq d(x, y) + d(y, z). \tag{10}$$

- Let $D(x, y)$ (and resp. $D(y, z)$) denote the set of nodes in D^{Ω} that are dominated by the node (x, y) (resp. (y, z)) in D^{Ω}.
- For each node (a, b) in D^{Ω}, if

$$u \in D(a, b) \cap D(x, y) \neq \varnothing, \quad v \in D(a, b) \cap D(y, z) \neq \varnothing,$$

then the constraint arising from the isolated triplet can be augmented by the following valid inequalities:

$$d(a, b) + d(x, z) \geq d(u) + d(v). \tag{11}$$

To see the validity of the above constraint, note that by the definition of domination, we have

$$d(x, y) \geq d(u), \quad d(y, z) \geq d(v), \quad d(a, b) \geq d(u), \quad d(a, b) \geq d(v).$$

If $d(a, b) = 0$, then $d(u) = d(v) = 0$ and, hence, (11) is trivially true. If $d(a, b) = 1$, then (11) follows from (10).

Three alternatives. We show that the polyhedron defined by (1), (2), and (5)–(8) need not be integral using a simple example. Let $\mathcal{A} = \{x, y, z\}$, and let

$$\Omega = \{xyz, yzx, zxy, xzy\}.$$

From (5)–(8), we get the following system of inequalities: $d(x, y) \leq d(x, z)$, $d(z, x) \leq d(y, x)$, $d(y, z) \leq d(y, x)$, $d(x, y) \leq d(z, y)$, $d(z, x) \leq d(z, y)$, $d(y, z) \leq d(x, z)$, $d(x, z) + d(z, y) \leq 1 + d(x, y)$, $d(y, z) + d(z, x) \geq d(y, x)$. A fractional extreme point of this system is

$$d(x, z) = d(y, x) = d(y, z) = d(z, x) = d(z, y) = 0.5; \quad d(x, y) = 0.$$

The only other fractional extreme point is:

$$d(x, y) = d(x, z) = d(y, z) = d(z, x) = d(z, y) = 0.5; \quad d(y, x) = 1.$$

We use the sequential lifting method to identify new valid inequalities from the isolated triplet (x, z, y). Consider the following set of inequalities:

$$1 + d(x, y) \geq d(x, z) + d(z, y). \tag{12}$$

Note that (x, z) dominates (y, z), and (z, y) dominates (z, x). We also have $d(y, x) \geq d(y, z)$, and $d(y, x) \geq d(z, x)$, and hence (y, x) dominates both (y, z) and (z, x). The sequential lifting method gives rise to

$$d(x, y) + d(y, x) \geq d(y, z) + d(z, x). \tag{13}$$

Also, for a pair of alternatives a and b, replacing $d(a, b)$ with $1 - d(b, a)$, results in another valid inequality, which we record as

$$d(x, y) + d(y, x) \leq d(z, y) + d(x, z). \tag{14}$$

More importantly, Equations (13) and (14) are *facets*. To see this, we first observe that the underlying polyhedron is full dimensional (dimension 6); and its extreme points are

$$\big\{e_4 = (0, 1, 0, 0, 0, 0),\ e_5 = (0, 1, 1, 1, 0, 0),\ e_6 = (1, 1, 0, 0, 0, 1),$$
$$e_7 = (1, 1, 1, 0, 1, 1),\ e_8 = (1, 1, 1, 1, 0, 1),\ e_9 = (1, 1, 1, 1, 1, 1)\big\},$$

where the components of each entry represent $d(x, y)$, $d(x, z)$, $d(y, x)$, $d(y, z)$, $d(z, x)$, and $d(z, y)$ (in that order). The elements e_1, e_2, e_3, e_4, e_5, and e_9 are affinely independent, and satisfy (13) as an equality; the elements e_1, e_3, e_5, e_7, e_8, and e_9 are affinely independent, and satisfy (14) as an equality. These two observations show, respectively, that Equations (13) and (14) are facets.

For $|\mathscr{A}| = 3$, we have enumerated all possible domains, and observed that the strengthened formulation using the sequential lifting method defines the convex hull of all ASWFs in each case. The same observation was obtained even for the case $|\mathscr{A}| = 4$. A natural question is if the sequential lifting method will yield all facets even for the case $|\mathscr{A}| \geq 5$.

7. Conclusions. In this paper, we study the connection between Arrow's Impossibility Theorem and IP. We show that the set of ASWFs can be expressed as integer solutions to a system of linear inequalities. Many of the well-known results connected to the Impossibility Theorem are direct consequences of the IP. Furthermore, the polyhedral structure of the IP formulation warrants further study in its own right. We have initiated the study on this class of polyhedra by characterizing the polyhedral structure of ASWFs on single-peaked domain. We have also demonstrated by an extensive computational experiment that the sequential lifting method proposed in this paper can be used to obtain the complete polyhedral description of ASWFs when the number of alternatives is small. Several interesting problems still remain:

1. Given a domain Ω specified by a certain membership oracle, is it possible to check for existence of nondictatorial ASWFs in polynomial time? Is the problem in the class NP?
2. The LP relaxation of our proposed IP formulation characterizes the ASWFs for single-peaked domain. What are the domains that can be characterized by the LP relaxation given by the sequential lifting method?
3. Can the conditions for ASCFs be written down as a system of integer linear inequalities?

We leave the above questions for future research.

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References

Arrow, K. J. 1963. *Social Choice and Individual Values*. Wiley, New York.

Dutta, B., M. Jackson, M. Le Breton. 2001. Strategic candidacy and voting procedures. *Econometrica* **69** 1013–1038.

Ehlers, L., T. Storcken. 2001. Arrow's Theorem in spatial environments.

Gibbard, A. 1973. Manipulation of voting schemes: A general result. *Econometrica* **41** 587–602.

Kalai, E., E. Muller. 1977. Characterization of domains admitting nondictatorial social welfare functions and nonmanipulable voting procedures. *J. Econom. Theory* **16**(2) 457–469.

Le Breton, M., J. Weymark. 1996. An introduction to Arrovian Social Welfare Functions on economic and political domains. *Collective Decision Making: Social Choice and Political Economy*. Norman Schofield, ed. Kluwer Academic Publishers, Boston, MA.

Maskin, E. 1995. Majority rule, social welfare functions, and game forms. K. Basu, P. K. Pattanaik, K. Suzumura, eds. *Choice, Welfare and Development.* Clarendon Press, Oxford, U.K.

May, K. O. 1952. A set of independent necessary and sufficient conditions for simple majority decision. *Econometrica* **20** 680–684.

Moulin, H. 1984. Generalized condorcet winners for single peaked and single plateau preferences. *Soc. Choice Welfare* **1** 127–147.

Muller, E. 1982. Graphs and anonymous social welfare functions. *Internat. Econom. Rev.* **23** 609–622.

———, M. Satterthwaite. 1977. The equivalence of strong positive association and strategy-proofness. *J. Econom. Theory* **14** 412–418.

Satterthwaite, M. 1975. Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *J. Econom. Theory* **10** 187–217.

Sen, A. 1966. A possibility theorem on majority decisions. *Econometrica* **34**(2) 491–499.

Sethuraman, J., C. P. Teo, R. Vohra. 2001. Integer programming and Arrovian Social Welfare Functions. Discussion paper 1316, The Center for Mathematical Studies in Economics and Management Science, Northwestern University, Evanston, IL.

Teo, C. P., J. Sethuraman. 1998. Geometry of fractional stable matchings and its applications. *Math. Oper. Res.* **23**(4) 874–891.

Wilson, R. B. 1972. Social choice theory without the Pareto principle. *J. Econom. Theory* **5** 478–486.

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